Discrete Groups and Visualization of Three-Dimensional Manifolds

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Abstract

We describe a software implementation for interactive visualization of a wide class of discrete groups. In addition to familiar Euclidean space, these groups act on the curved geometries of hyperbolic and spherical space. We construct easily computable models of our geometric spaces based on projective geometry; and establish algorithms for visualization of three-dimensional manifolds based upon the close connection between discrete groups and manifolds. We describe an object-oriented implementation of these concepts, and several novel visualization applications. As a visualization tool, this software breaks new ground in two directions: interactive exploration of curved spaces, and of topological manifolds modeled on these spaces. It establishes a generalization of the application of projective geometry to computer graphics, and lays the groundwork for visualization of spaces of non-constant curvature.

CR Categories and Subject Descriptors: I.3.3 [Picture/Image Generation] display algorithms I.3.5 [Computational Geometry and Object Modeling Graphics]: geometric algorithms, hierarchy and geometric transformations, I.3.7 [Three dimensional Graphics and Realism] color, shading, shadowing, and texture

Additional Key Words and Phrases: discrete group, tessellation, quotient space, projective geometry, hyperbolic geometry, spherical geometry, curvature, geodesic.

1 Discrete Groups

Symmetry, broadly speaking, implies a redundant supply of information. A mirror image contains the same information as the scene that it mirrors. The theory of discrete groups has been developed over the past 100 years as a formalization of the process of extracting a single copy of the information present in symmetric configurations. The discrete groups which we study here are groups of motions which act on a geometric space, such as Euclidean space, to produce tessellations by congruent non-overlapping cells. Familiar examples include wallpaper patterns, and the interlocking designs of M. C. Escher. We consider two simple examples before introducing mathematical definitions.

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1.1 The circle and the line

When we evaluate the expression $\sin(2\pi x)$ we are only interested in x mod 1, since sin is a periodic function: $\sin(2\pi x) = \sin(2\pi(x + k))$, where k is an integer. The set of all motions of the real line R by integer amounts forms a group Γ, which leaves invariant the function $\sin(2\pi x)$. We can form the *quotient* R/Γ, which is the set of equivalence classes with respect to this group. This quotient can be represented by the closed interval $[\mathbf{0}, \mathbf{1}]$, with the understanding that we identify the two endpoints. But identifying the two endpoints yields a circle. Once we know the values of $\sin(2\pi x)$ on the circle, we can compute it for any other value y, simply by subtracting or adding integers to y until the result lies in the range $[\mathbf{0}, \mathbf{1})$.

In this example the *discrete group* Γ is the set of transformations of R given by all translations $x \rightarrow x + k$, where k is an integer. Γ is *discrete* since no non-trivial sequence in Γ converges to the the identity element. The *quotient* of R under this action is S^1, the unit circle. We write $R/\Gamma = S^1$.

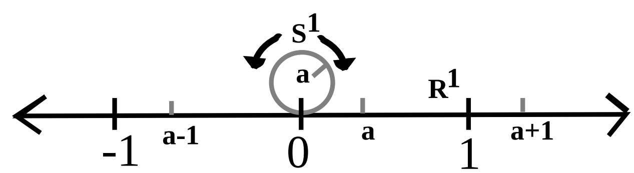

Figure 1: The circle is the quotient of R by the integers.

$I = [\mathbf{0}, \mathbf{1})$ is a *fundamental domain* for this group action. We can recover R from the fundamental domain and Γ: the union

$$\bigcup_{g\in\Gamma} gI$$

covers R without overlap.

We move into two dimensions to bring out other features of the concepts introduced in this example.

1.2 The torus and the plane

Instead of R we now work with R^2. Let Γ be the group of translations of R^2 generated by $(x, y) \rightarrow (x + 1, y)$ and $(x, y) \rightarrow (x, y + 1)$, that is, unit translations in the coordinate directions. What is the quotient R^2/Γ? Instead of the unit interval with its endpoints identified, we are led to a unit square that has its edges identified in pairs. If we imagine the square is made of rubber and that we can perform the identifications by bending the square and gluing, we find that the resulting surface is the torus T^2. See Figure 2.

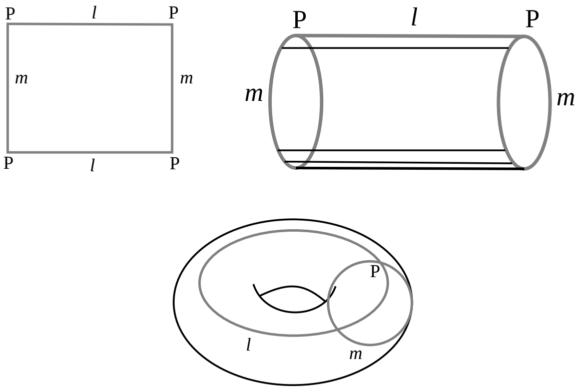

Figure 2: Making a torus from a square

1.3 Algebra and geometry: the fundamental group

A key element of this approach is the interplay of algebraic and geometric viewpoints. To clarify this, we introduce the *fundamental group* of a space, formed by taking all the closed paths based at some point P in the space. We get a group structure on this set: we can add paths by following one and then the other, and subtract by going around the second path in the reverse order. The zero-length path is the identity element. If one path can be moved or deformed to another path, the two paths correspond to the same group element. It is easy to check that different P's yield isomorphic groups. We say a space is *simply connected* if every closed path can be smoothly shrunk to a point, like a lasso, without leaving the space. [Mun75] The fundamental group of a simply connected space consists of just the identity element.

In the above example R^2 is simply connected; while T^2, the quotient, isn't. When X is the quotient of a simply connected space Y, we say that Y is the *universal covering space* of X. The importance of simply connected spaces in the study of discrete groups is due to a basic result of topology that (subject to technical constraints which we will consider satisfied) every space has a unique universal covering space [Mun75]. So in considering group actions, we need only consider actions on simply connected spaces.

The interplay of algebra and geometry reveals itself in the fact that the fundamental group of the quotient, a purely *topological* object, is isomorphic to the group of symmetries Γ, which arises in a purely *geometric* context.

1.4 Inside versus Outside Views

In the cases we will consider, the universal covering space X is a *geometric* space, that is, it comes equipped with a metric that determines distance between points and angles between tangent vectors. In this case we sometimes refer to X as a *model geometry*. This metric allows us to compute geodesics, or shortest paths, between points in the space [Car76]. The quotient space inherits this metric. R^2 is the universal covering space of T^2: if we unroll T^2 onto R^2, the copies of the torus will cover the plane completely, without overlap. We say these copies *tessellate* the plane. For some purposes the rolled-up torus sitting in R^3 is useful, but to gain the experience of what it is like to live *inside* the surface, we are better served by examining the *tessellation* of the universal covering space produced by the group.

For example, if we want to make pictures of what an inhabitant of T^2 sees, we will make them in R^2: Light follows geodesics, which appear to be very complicated on the rolled-up torus, but in R^2 are just ordinary straight lines. A complicated closed path based at P which wraps around the torus several times unrolls in the universal cover to be an ordinary straight line connecting P and hP for some $h \in \Gamma$. See Figure 3. An immediate consequence of this is that an observer on the torus based at P sees many copies of himself, one for every closed geodesic on the surface passing through P. For example, if he looks to the left he sees his right shoulder; if he looks straight ahead he sees his back. See [Wee85] for a complete and elementary description of this phenomenon. We say the rolled-up torus represents the *outsider's* view; while the unrolled view we term the *insider's* view, since it shows what someone living inside the space would see. The importance of the insider's view becomes more telling in three dimensional spaces, since to "roll up" our fundamental domains requires four or more dimensions. In this case the insider's view becomes a practical necessity.

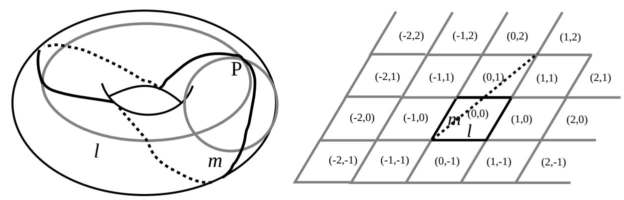

Figure 3: Outside and inside views of a complicated torus path

When we try to perform the analogous construction for the two-holed torus, instead of a square in the Euclidean plane R^2, we are led to a regular octagon in the hyperbolic plane H^2[FRC92]. We describe hyperbolic geometry in more detail below.

1.5 Definition of discrete group

A discrete group is a subgroup Γ of a continuous group G such that there is a neighborhood U of the identity in G with $U \cap \Gamma = I$, the identity element.

In the example of the torus above, the group Γ *acts* on R^2. Such an action on a topological space X is called *properly discontinuous* if for every closed and bounded subset K of X, the set of $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \phi$ is finite. In the cases to be discussed here, Γ is discrete if and only if the action of Γ is properly discontinuous.

If in addition the quotient space X/Γ is compact, we say that Γ is a *crystallographic*, or *crystal*, group.

The group of the torus discussed in 1.2 above is a crystallographic group, the simplest so-called *wallpaper* group. There are exactly 17 wallpaper groups of the Euclidean plane. See [Gun83] for a full discussion of this case and the details of a computer implementation.

1.6 Dirichlet domains

Given a discrete group, there is a technique for constructing a fundamental domain, known as a Dirichlet domain. We define it now for future reference. Given a discrete group Γ acting on a space X and a point $P \in X$, the orbit $O(P)$ of P under Γ is $\bigcup_{g \in \Gamma} gP$. Then the Dirichlet domain with respect to P is the set of points in X which are closer to P than to any other point of $O(P)$. We can be more precise. For each $Q \in O(P)$, construct the perpendicular bisector

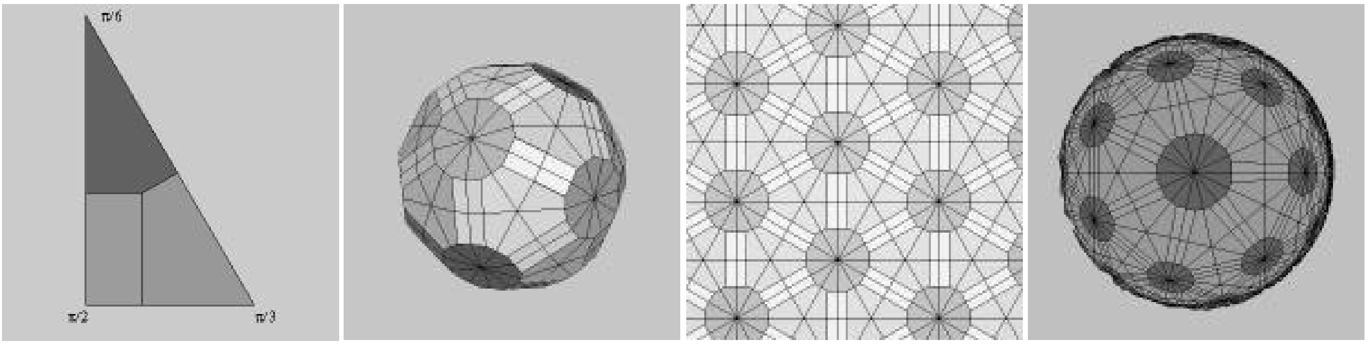

Figure 4: (235), (236) and (237) triangle groups tessellate S^2, R^2, and H^2.

M of the segment PQ. Denote by H_Q the half-space containing P bounded by M. Then the Dirichlet domain D_P determined by Γ and P is

$$\bigcap_{Q \in O(P)} H_Q$$

In practice, for many of the groups the intersection can be assumed to involve only finitely many H_Q's. The resulting polyhedron is convex. If a face F is determined by $g \in \Gamma$, then $g^{-1}F$ will be a congruent face F' determined by g^{-1}. This face pairing is used in the sequel. Note that, since D_P depends upon P, there are potentially many different shapes for the Dirichlet domain for a given group. [Bea83] Computational geometers may recognize that a Dirichlet domain with respect to P is a *Voronoi cell* with respect to the orbit of P.

2 Non-Euclidean Geometries

In the examples above, the model geometry was Euclidean. There are two other simply connected two-dimensional spaces in addition to R^2 which can serve as our model geometries: the sphere S^2 and the hyperbolic plane H^2. They have geometries (to be described in more detail below) which satisfy all the postulates of Euclidean geometry except for the Parallel Postulate: Given a line L and a point P not on L, there is a unique line M passing through P which is parallel to L. The sphere has no parallel lines; while H^2 has infinitely many for a given L and P. See [Cox65] for an account of the discovery and development of these non-Euclidean geometries.

An equivalent characterization of Euclidean, spherical, and hyperbolic geometry is that the sum of the angles of a triangle is, respectively, equal to, greater than, or less than, π. Figure 4 shows tessellations of these three spaces by triangles with angles $(\pi/2, \pi/3, \pi/n)$, where $n = (5, 6, 7)$ yields spherical, Euclidean, and hyperbolic space.

We now turn to demonstrating models for these three geometries which share a common root in projective geometry. This will lead directly to techniques for visualizing discrete groups which act on these spaces.

2.1 Projective geometry

Projective geometry is the geometry of lines without regard to distance or measure. It was discovered at roughly the same time as the the non-Euclidean geometries discussed above; we show in the sequel how it can be considered to be the fundamental geometry out of which the other geometries arise. [1]

The projective plane P^2 is gotten from the ordinary plane by adjoining a *line at infinity*. Projective space P^n can be constructed in every dimension n by adjoining an $n-1$ dimensional hyperplane at infinity. We assume the reader is familiar with homogeneous coordinates for projective space [Cox65]. The group of self-mappings of projective space P^n can then be represented via homogeneous coordinates as elements of the matrix group $PGL(R, n+1)$, the projective general linear group. This group consists of all invertible matrices of dimension $(n+1) \times (n+1)$, where two matrices are equivalent if one is a scalar multiple of the other [Cox87]. Much of the success of the approach described in this paper is due to the circumstance that many computer graphics rendering transformation pipelines support $PGL(R, 4)$.

2.2 From projective to metric geometry

Projective geometry does not include a notion of distance or angle measure. However, every projective transformation preserves a quantity known as the **cross ratio** . The cross ratio is a function of four collinear points:

$$\lambda(AB, CD) = \frac{(A-C)(B-D)}{(B-C)(A-D)}$$

Here the points are represented by a homogeneous coordinate system on their common line; for convienience we can assume this is ordinary Euclidean measure on the line. This invariant has been used by Cayley to construct metric geometries on the foundation of projective geometry [Cay59].

First choose a homogeneous conic Q which is to be invariant. The conic is known as the Absolute for the associated geometry. The projective transformations preserving Q form a subgroup H of the full projective group. Two given points P_0 and P_1 determine a line, which intersects the conic Q in a pair of points T_0 and T_1, whose coordinates may be complex numbers. Then define a distance function

$$d(P_0, P_1) = K \log \lambda(T_0 T_1, P_0 P_1) \quad (1)$$

where the constant K is determined according to the nature of Q in order to make the distance function real. Since the cross ratio is a multiplicative function, use of the log function yields an additive function. Measurement of angles between lines L_0 and L_1 proceeds

[1] See Appendix A.1

in like manner, by determining the two tangent lines to Q which lie in the pencil of lines determined by L_0 and L_1.

This yields models for spherical, hyperbolic, and Euclidean geometry which share the same straight lines; what is different is how distance along them and between them is measured. The subgroup H becomes the isometry group for the metric geometry.

We will for simplicity's sake work in two dimensions, that is, with homogeneous coordinates (x, y, w), and consider only distance measurement, not angle measurement. All our results generalize directly to arbitrary higher dimension. Since the cases of spherical and hyperbolic geometry are more straightforward, we begin with them.

2.2.1 Spherical geometry

For the spherical case, we choose Q to be the totally imaginary conic $x^2 + y^2 + w^2 = 0$. The proper choice for K is $i/2$. We can derive from Q an *inner product* between pairs of points: if $P_0 = (x_0, y_0, w_0)$ and $P_1 = (x_1, y_1, w_1)$ then $P_0.P_1 = x_0x_1 + y_0y_1 + w_0w_1$. Then (1) reduces to:

$$d(P_0, P_1) = \arccos\left(\frac{P_0.P_1}{\sqrt{(P_0.P_0)(P_1.P_1)}}\right)$$

This is the familiar measurement between points on the unit sphere. Projective transformations which preserve Q constitute the special orthogonal group SO(3), the group of rotations of three-dimensional Euclidean space. Although it is tempting to consider the familiar picture of S^2 sitting isometrically in R^3, it is more appropriate to think of the model presented purely in terms of P^2. In this model, to each point of P^2 we assign two antipodal points of S^2.

2.2.2 Hyperbolic geometry

For the hyperbolic case, we choose Q to be the totally real conic $x^2 + y^2 - w^2 = 0$, a cone aligned with the w-axis. The correct choice for K is $\frac{1}{2}$. The derived inner product of two points $P_0 = (x_0, y_0, w_0)$ and $P_1 = (x_1, y_1, w_1)$ is then $P_0.P_1 = x_0x_1 + y_0y_1 - w_0w_1$, sometimes called the Minkowski inner product. Our model for hyperbolic geometry will consist of the interior of this cone, where $P.P < 0$. Then (1) reduces to:

$$d(P_0, P_1) = \operatorname{arccosh}\left(\frac{P_0.P_1}{\sqrt{(P_0.P_0)(P_1.P_1)}}\right)$$

where P_0 and P_1 lie in the interior of the cone. The isometry group is SO(2,1), the so-called Minkowski group.

Consider the hyperboloid of two sheets H, defined by the condition $P.P = -1$. Just as the unit sphere is a model for spherical geometry, the upper sheet of H is a model for hyperbolic geometry. The most convenient model for H^2 is hidden within H. Consider the plane $w = 1$. It intersects Q in a circle that bounds a disk D. We can project our hyperboloid H onto D from the origin. This projection respects the distance function defined above (it is, after all, a projective invariant). Then D is a model of hyperbolic geometry, the so-called Klein or projective model. It is shown in the right-most figure in Figure 4. In three dimensions, this yields a model of H^3 as the interior of the unit ball in R^3. There are several other commonly used models of hyperbolic geometry, most notably the Poincaré or conformal model [Bea83]. Our choice of the projective model here was determined by the fact that it yields the correct results for visualizing the insider's view.

2.2.3 Euclidean geometry

Euclidean, or parabolic, geometry arises when apply a limiting process to the conic $\epsilon(x^2 + y^2) + w^2 = 0$. As $\epsilon \to 0$, the expression for distance reduces to

$$d(P_0, P_1) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$$

where P_0 and P_1 have been dehomogenized. The isometry group of this geometry $E(2)$ is the semi-direct product of $SO(2)$, the circle, and $R(2)$, the two-dimensional Euclidean translation group.

2.3 Comments

This development in terms of projective geometry is given fully in [Woo22] and is due to Cayley and Klein. For a treatment derived from the modern differential geometric viewpoint see [Car76]; for an implementation description following this viewpoint see [Gun92].

To justify the use of the names spherical and hyperbolic it is worthwhile to verify that the geometries induced by the indicated metrics on the indicated subspaces in fact yield geometries which behave correctly with respect to parallel lines and sums of angles of triangles.

For a detailed discussion of how to construct isometries of hyperbolic 3-space in the projective model discussed here see [PG92]. [2]

The above results, stated for the two-dimensional case, can be extended to arbitrary dimension.

3 Manifolds and Discrete Groups

An n-dimensional *manifold*, or n-manifold, is a topological space X such that X is locally homeomorphic to R^n, that is, every point of X has a neighborhood that can be mapped 1-1 and continuously onto a small ball in R^n. If in addition we can realize X as the quotient of a geometric space M by a discrete group, we say that X has a *geometric structure* modeled on M. A related concept to that of manifold is *orbifold*. An orbifold is like a manifold, but it may have singular points where it is locally homeomorphic not to R^n but rather to the quotient of R^n by a finite group. Orbifolds arise, generally speaking, when the elements of the discrete group have fixed points, such as rotations or reflections.

Initial work on the connection of discrete groups and theory of manifolds was done by Henri Poincaré in the 1880's. To this day much research in this field is driven by the Poincaré Conjecture, which asserts that a closed, connected, simply connected 3-dimensional manifold is homeomorphic to the 3-dimensional sphere S^3. This conjecture is closely related to the classification problem: making a list of all 3-manifolds. For example, in dimension 2, there is a uniformization theorem which says that any closed 2-dimensional manifold has a geometric structure modeled on one of S^2, R^2, or H^2. Recent work by Thurston and others has shown that many (possibly all) 3-manifolds have essentially unique geometric structures. That is, there are good reasons to believe that to every 3-manifold there corresponds an essentially unique discrete group [Thu82].

The geometric structures for 3-manifolds come from eight model geometries: R^3, S^3, and H^3 plus five additional simply connected spaces. The additional five are not as nice as the first

[2] See Appendix A.2.

three, since they are not *isotropic*: not all directions in space are the same. In any case, the most prevalent geometric structure is hyperbolic. The current software implementation does not support these five additional geometries.

In the discussion that follows, we will concentrate on the insider's, rather than the outsider's, view of three dimensional orbifolds. That is, we will look at the tessellations of the simply connected space (Euclidean, hyperbolic, or spherical) induced by discrete groups.

4 Software Implementation

4.1 OOGL

In order to visualize the spaces under consideration, we have developed an implementation within an object-oriented graphics library, OOGL. The generic OOGL class is `Geom`. Subclasses include include geometric primitives such as `PolyList`, `Vect`, `Bezier`, and `Mesh`; and organizational objects such as `List` and `Inst` (for instancing geometry). Methods with which `Geoms` come equipped include: `Bound, Create, Copy, Delete, Save, Load, Pick, and Draw .`

An interactive viewer, `Geomview` [MLP+], has been constructed based upon OOGL. It supports viewing in the three geometries discussed above: Euclidean, hyperbolic, and spherical. This is possible since as noted above isometries in the three geometries can be expressed as elements of $PGL(R, 4)$. The underlying low-level graphics libraries (in the case of OOGL, GL or Renderman [3]) support the use of elements of $PGL(R, 4)$ for modeling and viewing transformations. This is a result of the fact that $PGL(R, 4)$ is the smallest group which contains both the Euclidean isometries and the perspective transformation. The visualization task is also made easier by the fact that OOGL supports 4-dimensional vertices within all primitives. This provides a base for creating geometric models in hyperbolic and spherical space using homogeneous coordinates.

4.2 Shading

We have established how it is possible to implement non-Euclidean isometries using standard projective transformations. We have not addressed the question of correct lighting and shading of surfaces in these spaces. Indeed, the standard shading algorithms (in contrast to the standard transformations) are implicitly Euclidean. In order to model the behavior of light correctly in these non-Euclidean spaces, it is necessary to provide customized shaders which replace the default ones. This has been successfully achieved within the Renderman shading language [Ups89],[Gun92]. Figure 5 shows a view inside hyperbolic space from the movie "Not Knot". Interactive software shaders for OOGL for hyperbolic and spherical space have also been written.

These custom shaders use the expressions for distance and angle described in 2.2 to replace the Euclidean ones. Additionally, the decay of light intensity as a function of distance depends on the formula for the surface area of a sphere in each space. That is, the amount of light falling on an area element at distance d from a light source will be inversely proportional to the total area of the sphere with radius d. For example, in hyperbolic space light decays exponentially: the area of a sphere of radius r is given by $k\sinh(r)$

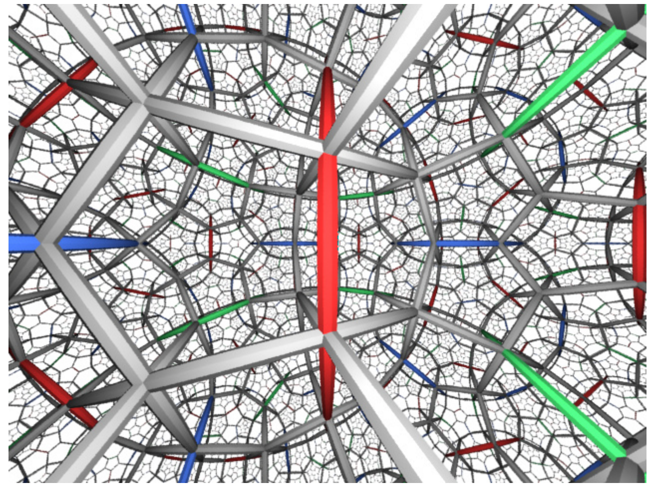

Figure 5: A view of the tessellation of hyperbolic space by regular right-angled dodecahedra, as in the movie "Not Knot". This image was rendered using Renderman.

and $\sinh(r) \approx \exp(r)$ for large r. The shaders used to create figures 6 and 9 also involve a term to model fog. [4]

5 The `DiscreteGroup` class

The `DiscreteGroup` class is a subclass of `Geom`. The minimal data includes a set of generating isometries represented by elements of $PGL(R, 4)$ and some geometric data, represented by other OOGL objects. The `DiscreteGroup` class supports the standard methods listed above, and other methods of its own.

Because of the close connection to manifolds outlined in Section 3, it can also be thought of as a `Manifold` class. Many design decisions were made to support visualization of the insider's view of a manifold. From this point of view, every element of the scene description belongs to the manifold and hence should be tessellated by the group in the process of creating the insider's view. We have departed from this philosophy in one important respect: we do not tessellate the lights contained in the scene description. To do so would have sacrificed interactivity for a questionable increase in authenticity.

Points of interest among `DiscreteGroup` methods include:

5.1 File format

There is an ascii file format for loading and saving discrete groups. [5] This format supports the three geometries described above, and includes lists of generators and group elements and also geometric objects for display within the tessellation.

5.2 `DiscreteGroupDraw`

Each `DiscreteGroup` instance includes a list of group elements and a collection of other `Geoms`. The general algorithm transforms each `Geom` by each group element and then draws it. There are some subtleties. Most of these groups are infinite, but we only compute and store a finite list of elements at any time. One of the difficulties

[3] GL is a trademark of Silicon Graphics, Inc.; and Renderman, of Pixar.

[4] See Appendix A.3.

[5] See Appendix A.4.

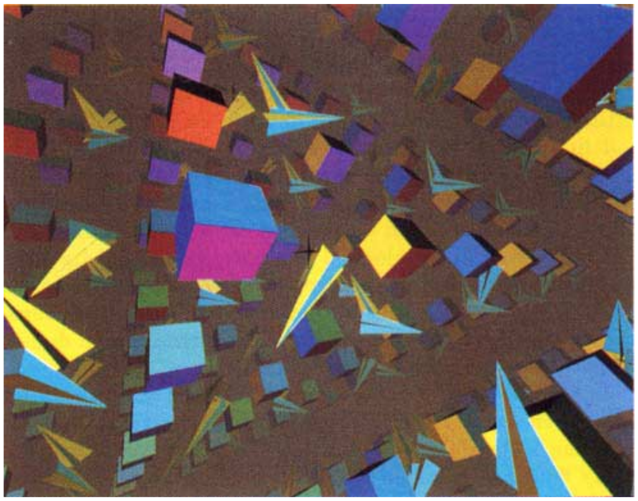

Figure 6: A view inside the Euclidean orbifold from “Not Knot” with the camera as a paper airplane.

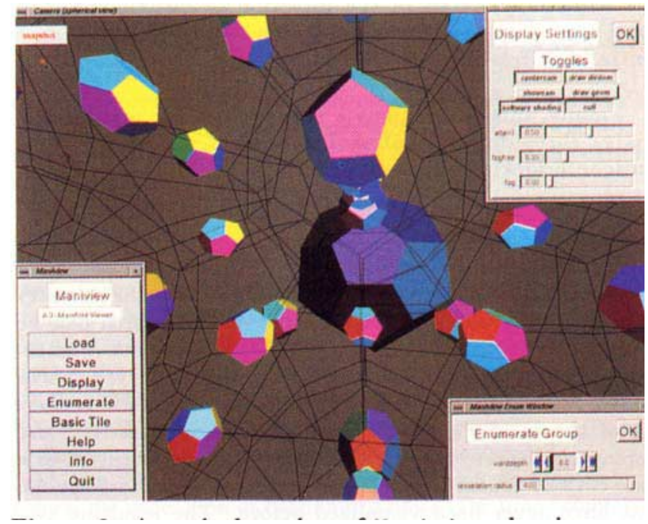

Figure 7: A typical session of `Maniview` showing some of its panels but hiding `Geomview` panels.

of navigating in the tessellations produced by discrete groups is that normal flight tends to wander to the edge of the computed tessellation. To solve this problem, the `DiscreteGroup` object is provided with an automatic centering mechanism. It detects when the camera leaves the Dirichlet domain defined by the group, and moves the camera by an isometry (determined by the face-pairings), to stay within this central region. Note that since lighting is not tessellated, lights must be defined within the camera coordinate system in order that lighting is invariant under this movement.

Another added feature is that there is a separate associated `Geom` which represents the camera, or observer. Before being tessellated it is moved to the location of the camera, which as described above is constrained to stay within the Dirichlet domain. The observer then becomes aware of his own movement in the space. This is an important feature especially for detecting the singular locus of orbifolds. For example, when the camera approaches a axis of symmetry of order n in an orbifold, this fact is made clear by the approach of $n-1$ other copies of the camera to the same axis, a symmetry which the geometry of the Dirichlet domain alone may not reveal.

5.3 `DiscreteGroupEnum(int constraint() )`

is a method for enumerating lists of group elements given the generators. One such list is used by the draw routine: it defines which copies of the fundamental domain to draw. The constraint function accepts a single group element and returns 0 or 1 according to whether it satisfies its criteria. For example, a matrix may be rejected if it moves the origin far, its determinant is small, or its expression as a word in the generating elements is long. This enumeration software uses software acceleration provided by the theory of automatic groups [ECH$^+$91], [Lev92] if an automatic structure has been provided for the discrete group.

5.4 `DiscreteGroupDirDom`

creates a fundamental domain using the Dirichlet domain algorithm described above. This is useful for exploring groups for which no other geometry has been provided. For display purposes, both a wire-frame of the full polyhedron and a possibly scaled version with faces colored to reflect the face-pairing identities are drawn. See Figure 9. The user can deduce features of the group by examining the face-pairing patterns, or by moving the distinguished point P.

6 Example applications

A variety of applications have been developed based on the `DiscreteGroup` software class.

`Maniview` is short for *Manifold Viewer*. In the paradigm of object-oriented software tools, it is essentially an Inspector for the class `DiscreteGroup` . `Maniview` communicates with `Geomview` via a two-way pipe. `Geomview` reads the description of the discrete group output by `Maniview` and displays it. The user typically loads a discrete group into `Maniview`, and then manipulates the discrete group via a set of control panels. These panels are grouped into: display settings, enumeration of group elements, choice of fundamental tile, and saving and loading various elements. A typical snapshot of a `Maniview` session is shown in Figure 7.

One of the milestones in the theory of discrete groups was the enumeration of the 230 crystal groups in three dimensional Euclidean space at the end of the nineteenth century. For a survey see [LM78],[Sch80]. `eucsyms`, an interactive application which allows the exploration of these groups has been developed by Olaf Holt at the Geometry Center, and adapted to use the `DiscreteGroup` software. `eucsyms` is connected by a two-way pipe with `Maniview`. Euclidean crystal groups can be written as a semi-direct product of a translation subgroup and a finite subgroup of SO(3), a *point* group. The structure of `eucsyms` reflects

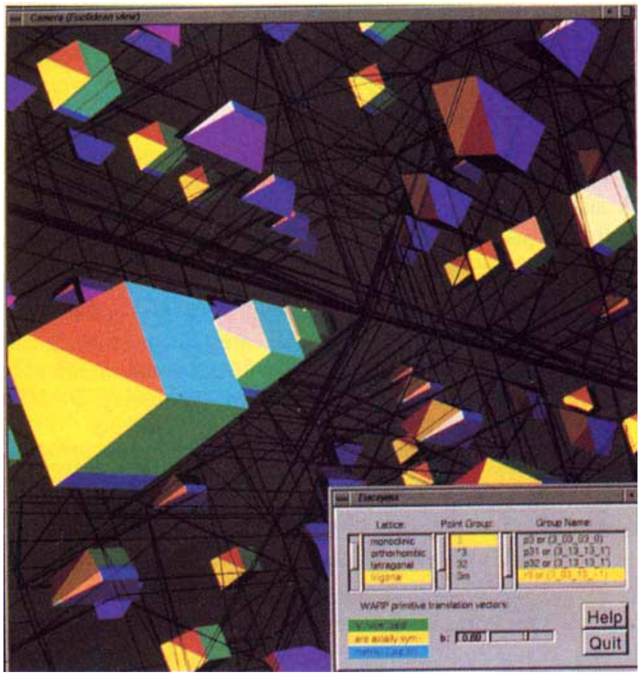

Figure 8: A snapshot of a session using `eucsyms`, an application for exploring the 230 Euclidean crystal groups. *This figure shows an earlier version of eucsyms than that described in the paper. See the paper version for an up-to-date figure.*

this decomposition. Choice of group proceeds by first choosing the lattice which the translation subgroup leaves invariant (one of seven lattice types), then choosing the point group, finally choosing the particular semi-direct product of the two. Figure 8 shows a view inside the symmetry group *r3*.

We have also hooked up `Maniview` to a powerful program for computing hyperbolic structures on three dimensional manifolds, `snappea` by Jeff Weeks [Wee]. This is a popular tool used by research topologists to construct and examine three dimensional manifolds.

`Geomview`, `Maniview`, `eucsyms`, and `snappea` are all available via anonymous ftp from geom.umn.edu [128.101.25.35]. Some of the computation of the groups and geometrical models shown in the figures have been computed using a Mathematica [6] package developed at the Geometry Center, also available via anonymous ftp from the same site.

7 Example spaces

7.1 "Not Knot"

The mathematical animation "Not Knot" [GM91] pioneered the visualization of the insider's view of hyperbolic space. It features one Euclidean orbifold (see Figure 6) and a series of hyperbolic orbifolds converging to a hyperbolic manifold that is the complement of the three linked circles known as the Borromean rings. Figure 5 shows one of these orbifolds, which tessellates H^3 with right-angled dodecahedra. One of the six generators is a rotation of $\frac{\pi}{2}$ around

[6] Mathematica is a trademark of Wolfram Research, Inc

Figure 9: A view of the 120-cell. We reduce the size of the shaded polyhedra but draw the original edges.

the large red axis. As a matrix this generator is:

$$\begin{pmatrix} -1.618033 & 1.618033 & 0 & -2.058171 \\ -1.618033 & 0 & 0 & -1.272019 \\ 0 & 0 & 1. & 0 \\ 2.058171 & -1.272019 & 0 & 2.618033 \end{pmatrix}$$

Note that all the non-zero entries are powers of the golden ratio. This is an example of an *arithmetic* group and is of particular mathematical interest.

The discrete groups underlying "Not Knot" have been converted into the `DiscreteGroup` format. Now, viewers interested in exploring the spaces depicted in "Not Knot" can do so.

7.2 The Poincare homology sphere

Possibly the most famous three dimensional spherical manifold is the so-called Poincare homology sphere. It arises abstractly by identifying the opposite faces of a regular dodecahedron with a twist of $\pi/5$. [7]

The tessellation of S^3 corresponding to this manifold consists of 120 regular dodecahedra, which meet 3 around each edge, and is known as the 120-cell or dodecahedral honeycomb [Cox73]. In contrast to the right-angled dodecahedron of hyperbolic space, these dodecahedra have dihedral angles of $\frac{2\pi}{3}$.

An inside view of this manifold appears in Figure 9. Note that the largest dodecahedron, which completely fills the view as if it surrounds the viewer, is also the farthest away. This is a typical feature of life in spherical space; as objects move away they decrease in size until they reach a maximum distance of $\pi/2$, then they begin to increase in size until they reach the antipodal point of the viewer at a distance of π, where they expand to fill completely the field of view, since every geodesic leaving the observer also passes through the antipodal point. Stereo viewing in spherical space would place great strain on Euclidean trained eyes: when an object is exactly at the equator, the lines of sight from an observer's eyes are parallel;

[7] See Appendix A.5.

as an object moves beyond the equator, the observer must look "anti-crosseyed" at it.

8 Directions for further work

Common ancestry in projective geometry means that some important procedures can be shared with traditional Euclidean systems. However, there remain a host of computer graphics issues related to modeling and animation in non-Euclidean spaces to be addressed. Many geometric constructions are very different. For example, consider a equidistant curve, that is, the set of points equidistant from a line. In the Euclidean plane an equidistant curve is a parallel line. But equidistant curves in spherical and hyperbolic space are not straight lines. What, then, is the proper generalization of a cylinder in these spaces? Also, neither space allows similarity transformations: changing the size of an object changes its shape! Other questions arise. What sort of harmonic analysis is available to synthesize fractal terrains and textures in these spaces? If we hope to do physically-based modeling in these spaces, we need to expand our understanding of the laws of physics beyond the behavior of light described above in relation to shading. Finally, the theory of splines in non-Euclidean spaces was explored in [GK85].

In the area of topological content, one obvious goal is to implement the five non-isotropic three dimensional model geometries. Also, there are many sorts of discrete groups, particularly those that create fractal patterns, which do not fit neatly into the current framework.

In the direction of mathematical research and user interface, the efforts described here suggest various techniques for exploring 3-manifolds. Connecting this software with virtual reality technology would allow the researcher to perform a variety of explorations of the space. The use of sound also promises to yield useful evidence.

Looking at the wider world of Riemannian geometry, this work is one step in the direction of visualizing arbitrary curved spaces, the Riemannian manifolds that figure centrally in relativity and cosmology. For related work see [HD89].

Finally, this work opens a new domain for artistic creativity, three dimensional analogues of M. C. Escher's dramatic interlocking planar tessellations.

9 Conclusions

Approaching metric geometries via their common ancestry in projective geometry yields simple models which can be directly implemented in existing rendering systems. The resulting systems allow interactive navigation of curved spaces for the first time. Custom shaders provide realistic rendering of the insider's view. Methods for manipulating and displaying discrete groups allow interactive exploration of a wide class of topological manifolds modeled on these spaces, that have never been visualized before. The resulting system provides a unique tool for mathematicians, educators, and scientists and artists whose work is related to spatial symmetry.

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A Appendices for CD-ROM

The following sections were deleted from the printed version of the paper due to space limitations that are not present in this CD-ROM version.

A.1 More on projective geometry

The main contributors to projective geometry include Poncelet, von Staudt, Cayley, and Klein [Boy68].

Projective geometry is distinguished by a perfect duality between point and line; every two points determine a line; but equally surely, every two lines determine a point, the point of intersection. "Parallel" lines also intersect in a point, a *point at infinity*. Consequently, every theorem has a dual version in which point and line are reversed. Projective space P^n can be constructed in every dimension n by adjoining an $n-1$ dimensional hyperplane at infinity. After we adjoin these points, they have no special status. A full account can be found in [Cox65].

Coordinates in the projective plane arise from the choice of a triangle of reference. Then every point can be written as a weighted sum $(xX + yY + wW)$ where X,Y,W are the vertices of the triangle of reference, and its coordinates are the vector (x, y, w). By definition, $(\lambda x, \lambda y, \lambda w) = (x, y, w)$ for any non-zero λ. If the triangle of reference is chosen to include the Cartesian origin as one vertex, the line at infinity as the opposite side, and the other two sides are perpendicular to each other, we arrive at *homogeneous coordinates* for the Euclidean plane, in which we can choose λ so that $w = 1$ except for points on the line at infinity. The resulting (x, y) coordinates are the familiar Cartesian coordinates [Woo22].

A.2 Constructing Isometries

Note that it is possible to express the Absolute Q as a symmetric matrix. For example, for the hyperbolic plane

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then the Minkowski inner product $P_0 . P_1 = P_0 Q P_1^t$, where P_1^t is the column vector form of P_1.

All isometries in these geometries may be constructed as products of reflections [Cox65]. We consider a reflection in a hyperplane L in H^n. This hyperplane determines a 1-dimensional orthogonal subspace N(orthogonal with respect to the Minkowski inner product!). After normalizing N, the reflection in L acting on a point x can be expressed as $x - 2(x.N)N$. We can write this as a matrix, $x' = xA$ with $A_{ij} = I_{ij} - 2N_i Q_{ii} N_j$ where Q is the matrix form of the Absolute. Similar expressions apply for reflections in R^n and S^n.

A.3 Spherical space caveats

There are some details to the implementation of visualization in S^3 which deserve mention. They reflect the fact that each point of P^n corresponds to a pair of antipodal points on S^n. The behavior of the rendering system in this situation depends on how the conversion from homogeneous coordinates into 3D is done. In some systems, points with $w < 0$ are negated to force $w \geq 0$, then clipping is performed. This essentially collapses S^3 onto P^3. The other alternative is that the region $w < 0$ is treated separately from

$w \geq 0$; typically, clipping is done so that line segments are clipped to lie within the latter region. This is preferable for visualizing S^3. The result is that half the sphere, where $w < 0$, is invisible at any moment. This can be arranged to be the hemisphere that is behind the camera. See [FvDFH90] for a discussion of homogeneous clipping.

A.4 Sample data file

The following is a sample `DiscreteGroup` data file `borrom2.dgp`, which describes the Euclidean orbifold shown in Figure 6.

```
# Comments are delimited by '#'s.
DISCGRP                          # Class Identifier
(group borrom2 )                 # Group name
(comment "  Order 2 Borromean orbifold.  " )        # Arbitrary comment
(attribute Euclidean )           # { Euclidean | hyperbolic | spherical }
(enumdepth 4 )                   # Length of words in generators to compute
(enumdist 10.0 )                 # No group element that moves origin > 10
(dimn 3 )                        # Dimension of the underlying space
(ngens 6 )                       # Number of generators
(gens                            # List of generators with symbolic names
# The generators are 6 180 degree rotations around axes lying on centers
# of the sides of a cube of side-length 1.0.
# a is rotation around line parallel to z-axis, passing through (.5, 0, 0).
a
-1 0 0 0
0 -1 0 0
0 0 1 0
1 0 0 1
# ditto for d, through (-.5, 0, 0)
d
-1 0 0 0
0 -1 0 0
0 0 1 0
-1 0 0 1
# b rotates a line parallel to x-axis, through point (0, .5, 0)
b
1 0 0 0
0 -1 0 0
0 0 -1 0
0 1 0 1
# ditto for e, through (0, -.5, 0)
e
1 0 0 0
0 -1 0 0
0 0 -1 0
0 -1 0 1
# etc.
c
-1 0 0 0
0 1 0 0
0 0 -1 0
0 0 1 1

f
-1 0 0 0
0 1 0 0
0 0 -1 0
0 0 -1 1

)
# distinguished point for Dirichlet domain computation
(cpoint 0.000000 0.000000 0.000000 1.000000 )
# geometry to use to represent the camera: a paper airplane
(camgeom
{ = OFF                 # see man 5 OOGL for this file format
5 2 5

0 0 0
-0.1 0 0.5
0.1 0 0.5
0 -0.1 0.5
0 0.1 0.5

3         0 1 2         200 200 0 .8
3         0 3 4         0 200 200 .8
}
)                 # end of DiscreteGroup format file
```

A.5 More on the 120-cell

Each of its six generators moves a face of one dodecahedra so it lines up with its opposite face; a typical generator looks like:

$$\begin{pmatrix} 0.809017 & 0.5 & 0 & 0.309017 \\ -0.5 & 0.809017 & 0.309017 & 0 \\ 0 & -0.309017 & 0.809017 & 0.5 \\ -0.309017 & 0 & -0.5 & 0.809017 \end{pmatrix}$$

Its first homology group is trivial, hence the name. (The first homology group is a commutative version of the fundamental group.) Originally Poincaŕe had stated his famous conjecture in terms of the first homology groups, and this manifold had provided the important counterexample which led to the revised conjecture into its current form.

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