INTERNAL SYMMETRY GROUPS OF QUANTUM GEONS

John L. FRIEDMAN [1] and Donald M. WITT
Department of Physics, University of Wisconsin, Milwaukee, WI 53201, USA

Received 6 July 1982

In canonical quantum gravity asymptotically trivial diffeomorphisms not deformable to the idendity can act nontrivially on the quantum state space. We show that for many 3-manifolds, the inequivalent diffeomorphisms comprise coverings in SU(2) of crystallographic groups. When the diffeomorphism $R_{2\pi}$ associated with 2π-rotation is nontrivial, state vectors can have half-integral angular momentum; we list all 3-manifolds with $R_{2\pi}$ trivial.

In classical gravity, two metrics are physically equivalent if they differ only by a diffeomorphism. Within the canonical framework of quantum gravity, however, state vectors need be invariant only under diffeomorphisms (hereafter, "diffeos") that are continuously deformable to the identity [1–4] and which, for asymptotically flat spacetimes, are trivial at infinity. Remarkably, because the diffeo associated with a 2π-rotation, although trivial at infinity, is not in general deformable to the identity [1,2], states with half-integral angular momentum can arise in quantum gravity. More generally, if D_M is the group of asymptotically trivial diffeos of a manifold M, the group $G \equiv \pi_0(D_M)$ of inequivalent diffeos is a symmetry group acting on states associated with M.

In the geometrodynamics picture of a topological structure as a particle – a "geon" – G is an internal symmetry group: a group whose elements commute with but do not include the external symmetries (rotations and translations) and (as long as the topology is fixed) with time evolution. We use the term geon to mean a *prime factor* of the manifold, a region of nontrivial topology which cannot be subdivided by a sphere into two disjoint regions, each with nontrivial topology [‡1]. The space $\mathscr{H}_M$ of state vectors corresponding to a single geon consists, in the Schrödinger representation, of wave functions with support on metrics on a fixed prime manifold, M. $\mathscr{H}_M$ can be decomposed into irreducible subspace under the action of G, and the irreducible representations of G may be regarded as the possible particle multiplets associated with a gluon [2,5]. Every prime 3-manifold N can be obtained from a solid polyhedron by identifying faces [6]. The corresponding noncompact space $M = \mathbf{R}^3 \# N$ is obtained by removing a polyhedron from $\mathbf{R}^3$ and similarly identifying faces of the boundary of the hole. We will see that at least for prime manifolds known to arise in classical relativity, the symmetry group G is often isomorphic to the covering group in SU(2) of the polyhedron's symmetry group.

In Yang–Mills quantum theory, the state space is similarly invariant only under gauge transformations that are deformable to the identity [7,8] and which are trivial at infinity. There, however, the classes of inequivalent gauge transformations form an abelian group isomorphic to the additive integers $\mathbf{Z}$, and so the group has only one-dimensional representations, each

[1] Research supported in part by the National Science Foundation.

[‡1] Footnote, see next column.

[‡1] On non-orientable manifolds with handles, there is an ambiguity in stating what the prime factors of a manifold are, because one can convert an orientable handle into a non-orientable one by sliding one end of it through a non-orientable prime factor. Here, however, one can resolve the ambiguity by using "geon" to refer to a prime factor *metrically* separated from all other prime factors. (Note that handles are here regarded as single geons, corresponding to the physical assumption that prime factors will be microscopic – that in particular, the ends of a handle will have small separation.)

fixed by a single angle θ: $g_n\psi = e^{in\theta}\psi$, where g_n is a gauge transformation of winding number n [n is the winding number of g_n regarded as a map from $S^3 \approx$ (euclidean space with a point at infinity) to the gauge group]. In gravity, because the corresponding group G is in general not abelian, its irreducible representations are in general not all one dimensional. Isham [3,4] regards the possible "θ-states" of gravity as the *characters* of G; as in the Yang–Mills case, however, *all* irreducible representations can be built from a set of states of the form $\chi_i\psi$, where ψ is a wave function centered about a particular 3-geometry (M, [g]), where the χ_i are inequivalent diffeos of G and $\chi_i\psi(g) = \psi(\chi_i^{-1}g)$; and where [$g$] is an equivalence class of metrics with $[g] = [g'] \Leftrightarrow g' = \chi g$ for some diffeo χ deformable to the identity ‡2.

Symmetry groups of compact spaces. Compact 3-manifolds can be decomposed into a direct sum ‡3 of prime factors and (to within standard conjectures of topology)‡4 all prime orientable 3-manifolds are one of the following types [10]: (i) Spherical spaces. These are manifolds of the form S^3/H, where H is a finite subgroup of SO(4) acting freely on S^3. (ii) $S^1 \times S^2$ (a "handle"). (iii) K(Π, 1)'s, manifolds of the form $\mathbf{R}^3/\Pi$ where Π is a group of homeomorphisms acting freely on $\mathbf{R}^3$. Prime non-orientable manifolds are either $S^1 \underset{\sim}{\times} S^2$ (twisted handle), $S^1 \times P^2$ (P^2 the projective plane), $\widetilde{K}(\Pi, 1)$'s, or manifolds whose orientable double covering is not prime.

All geodesically complete manifolds M admitting asymptotically flat 3-metrics with a single asymptotic region have the form of a compact manifold with a point (the point i_0 "at infinity") removed: $M \approx N\backslash\{i_0\} \approx \mathbf{R}^3 \# N$. In classical relativity it may be that the K(Π, 1)'s do not occur in asymptotically flat vacuum spacetimes ‡5 but all the remaining manifolds do, and these are completely classified, since the finite subgroups of SO(4) are well known. We will first consider the symmetry groups of compact spaces N and then turn to the more interesting case of the manifolds $M = \mathbf{R}^3 \# N$ which can arise as asymptotically flat spaces whose symmetry groups can be regarded as internal symmetries of geons.

There is a standard conjecture [13] that for spherical spaces, the homotopy groups of the diffeos are the homotopy groups of the isometries, and for most spherical spaces, this has been proved (in many cases quite recently [14,15]) for the group π_0(D). Now the isometries of S^3/H are those elements g of SO(4) for which the image of an orbit of H is another orbit of H, and this is true precisely when $gHg^{-1} = H$, when g is in the normalizer N(H) of H. The conjecture is then

$$\pi_0(D) = \pi_0[N(H)/H], \tag{1}$$

where "N(H)/H" appears instead of "N(H)" because any $h \in H$ acts as the identity on S^3/H.

We list below (in table 1) the spherical spaces [16] and their symmetry groups π_0(D), assuming them to be given by eq. (1). Those for which (1) has not been proved are marked by a dagger. The first set of groups listed are also subgroups of SU(2). Here T^*, O^* and I^* denote the covering groups in SU(2) of the tetrahedral, octahedral and icosahedral subgroups of SO(3). Similarly, for each $m \geq 2$, D_m^* is the covering group in SU(2) of the dihedral group of order $2m$, the symmetry group in SO(3) of the m sided prism. These spaces are obtained [17] by identifying faces of polyhedra and the names of the spaces refer to these polyhedra.

For the handle $N = S^1 \times S^2$, $\pi_0(D) = Z_2 \times Z_2 \times Z_2$ [18]; for the nonorientable handle $S^1 \underset{\sim}{\times} S^2$ [19] and for $S^1 \times P^2$, $\pi_0(D) = Z_2 \times Z_2$. Finally, for the K(Π,1)'s, π_0(D) is known when the space is "sufficiently large" [16] (which for these spaces is equivalent to requiring that the first homology group be infinite). Then π_0(D) = Out(Π), the outer automorphisms of the group Π.

‡2 More recent work by Isham [9] also recognizes the general irreducible representations.

‡3 If A and B are manifolds, the direct sum A # B is obtained by removing a ball from each manifold and then identifying the spherical boundaries. In two dimensions, for example (where a ball is a disk), the direct sum of a plane and two tori is a plane with two handles on it.

‡4 One needs the Poincare conjecture to know that the sphere is the only compact, simply-connected 3-manifold, and the Smale conjecture (whose proof is claimed by Hatcher) to know that the only groups of homeomorphisms H acting freely on S^3 are isomorphic to subgroups of SO(4).

‡5 Shoen and Yau [11] claim that if on $\mathbf{R}^3$ #N, there is initial data satisfying the vacuum constraint equations (or the constraint equations with matter satisfying a local energy condition) then N admits a metric with positive scalar curvature; Gromov and Lawson [12] claim that the K(Π, 1)'s have no metrics with positive scalar curvature. (However, D. Brill now claims a counterexample to the Schoen and Yau assertion when matter is present.)

Table 1
Compact spherical spaces: $N = S^3/H$, where $H \subset SO(4)$.

H	Name of space	$\pi_0(D)$
T^*	octahedral space	$Z_2{}^\dagger$
O^*	truncated cube space	1
I^*	dodecahedral space	$1^\dagger$
D^*_m	prism manifolds	P_3, (permutation group of 3 objects), m=2 Z_2, $m \geqslant 3$
$H \times Z_p$	(H is any one of the groups listed above and Z_p a cyclic group of relatively prime order)	$\pi_0[D(S^3/H)] \times Z_2$
Z_p	lens spaces $L(p,q)$	Z_2, if $q^2 \not\equiv \pm 1 \bmod p$, otherwise, $\pi_0(D)$ is Z_2, if $q \equiv \pm 1 \bmod p$ Z_4, if $q^2 \equiv -1 \bmod p$ and $q \not\equiv \pm 1 \bmod p$ $Z_2 \times Z_2$, if $q^2 \equiv 1 \bmod p$ and $q \equiv \pm 1 \bmod p$
$D'_{2^k \cdot (2n+1)}$	prism manifolds	$Z_2 \times Z_2$
$T'_{8 \cdot 3^k}$	–	$Z_2{}^\dagger$
$D' \times Z_p$, $T' \times Z_p$		$Z_2 \times Z_2$, Z_2

Symmetry groups of prime, asymptotically flat spaces. The corresponding open manifolds, M, obtained by removing a point from S^3/H, can be more picturesquely described as the result of removing from $\mathbf{R}^3$ the same polyhedron and then identifying the boundaries of the resulting hole. The symmetry group $G = \pi_0(D_M)$ (where D_M is the group of diffeos of M that are trivial at infinity) is in general the symmetry group of the polyhedron – or its covering group in SU(2). Representative diffeos in each class can be described as follows. Let $R(\theta_0\hat{n})$ be a rotation of $\mathbf{R}^3$ that corresponds to a polyhedral symmetry and M the manifold constructed by identifying faces of the polyhedron cut from $\mathbf{R}^3$. Then an associated asymptotically trivial homeomorphism of M can be written in the form $R[\theta(\lambda)\hat{n}]$, where $\theta(0) = \theta_0, \theta(\infty) = 0$ and λ, $0 \leqslant \lambda < \infty$, is a scalar that labels concentric polyhedra.

Rotations, in contrast, have the form $\exp(\phi\boldsymbol{\xi})$, where the vector field $\boldsymbol{\xi}$ generates a rotational subgroup of the symmetry group at spatial infinity. The 2π-rotation, however, is trivial at infinity, and so, as mentioned previously belongs to the internal symmetry group G. When the 2π-rotation is a nonzero element of G (so that a 2π-twist near infinity cannot be untwisted by communicating it to the interior), the space of state vectors ψ satisfying the momentum constraint includes states of half-integral angular momentum [1,2]. In the cases we have calculated, the internal symmetry group G is then the covering group in SU(2) to a crystallographic symmetry group and G may be said to mix internal and external symmetries. (When the 2π-rotation is deformable to the identity, the internal and external symmetry groups have only the identity element in common.)

To find the symmetry groups for these non-compact spaces, we use the fact that a diffeo trivial at infinity for the non-compact space M is equivalent to a diffeo that fixes a point i_0 and a frame in the compactified $N = M \cup \{i_0\}$. We will give an explicit example of the computation and then list the groups G obtained in this way. Let D_N be the group of diffeos of N, and let D_M be the group of diffeos of M that are the identity at infinity. Denote by e_0 a frame (triad) at i_0, by χ a diffeo of D_N, and by χe_0 the frame at $\chi(i_0)$ obtained by dragging e_0 to i. Finally let E be the bundle of right handed frames of N. Then D_N can be given the structure of a bundle over E_N with fiber isomorphic to D_M as follows.

Define a projection p: $D_N \to N$ by $p(\chi) = (\chi(i_0), \chi e_0)$. That is, a diffeo χ is in the fiber over (i, e) if χ takes (i_0, e_0) to (i, e). Two diffeos over (i, e) are related by a diffeo that fixes (i, e) and thus the fibers are isomorphic to D_M. The homotopy exact sequence for fiber bundles [20] has the form [‡6]

$$\ldots \to \pi_1(D_N) \xrightarrow{\alpha} \pi_1(E) \xrightarrow{\beta} G \xrightarrow{\gamma} \pi_0(D_N) \to 1 .$$

For $N = S^3/H$, $\pi_1(N) = H$ and $o(\pi_1(E)) = 2\, o(H)$, where "o(H)" means the order of (number of elements in) H. Exactness implies $o[G] = o[\pi_0(D_N)].o(\mathrm{Im}\,\beta)$ and $o[\pi_1(E)] = o(\mathrm{Im}\,\alpha)\cdot o(\mathrm{Im}\,\beta)$. Hence

$$o[G] = 2\, o[\pi_0(D_N)]\ o(H)/o(\mathrm{Im}\,\alpha). \qquad (2)$$

Consider now the case $H = I^*$, the 120 element covering group to the icosahedral group. It is not difficult to verify that α has nontrivial image, so $o(\mathrm{Im}\,\alpha) \geqslant 2$; and from table 1, $o[\pi_0(D_N)] = 1$. Then (2) implies $o(G) \leqslant 120$. But for the dodecahedral space, $N = S^3/I^*$, every symmetry of the dodecahedron gives rise to a nontrivial element of $\pi_0(D_M)$: the diffeos described in the first paragraph of this section act nontrivially on $\pi_1(M) = I^*$. Finally, the rotation $R_{2\pi}$ is nontrivial in G and its composition with each of the polyhedral diffeos is a distinct element of G. Because the diffeos enumerated thus far form a 120 element group isomorphic to I^*, $I^* \subset G$; and (2) then implies that $I^* = G$. In this way we obtain G for the spherical spaces in table 2.

For the handle and twisted handle, G is given, for example, in [19]; in both cases the diffeos are a 2π-twist between two spheres surrounding a single end and an exchange of the ends. (The additional Z_2 for $S^1 \times S^2$ in the compact case comes from the reflection $\theta \to -\theta$, where θ parametrizes the S^1, but this is not trivial at infinity and so is not in G). Because no $K(\Pi, 1)$ has finite fundamental group, $R_{2\pi}$ is not trivial [1,21] and $\{R_{2\pi}, 1\}$ is a Z_2 subgroup of G. Modulo $R_{2\pi}$, G is isomorphic to the group of orientation preserving automorphisms of Π [22] when $K(\Pi, 1)$ is sufficient large: $G/Z_2 \approx \mathrm{Aut}^+(\Pi)$.

Several geons. The symmetry group G for the connected sum $M = \mathbf{R}^3 \# N_1 \# \ldots \# N_p$ of p prime factors, although not simply the direct product of the internal symmetry groups of each price factor, is easily described. The additional diffeos are generated by interchanges of identical factors and by slides [21] – diffeos that move a prime factor or one end of a handle around a nontrivial loop of the surrounding manifold [‡7]. To each prime factor N_i (or each end of a handle) and each class of loops in $M \setminus N_i$ corresponds a slide; and the sub-

‡6 The extension of the exact sequence to π_0(fiber) $\to \pi_0$ (bundle) $\to 1$ holds when, as here, the fiber and bundle spaces are topological groups and the base space is connected.

‡7 Let B be a ball containing a prime factor N of M and let $C(\theta)$, $0 \leqslant \theta \leqslant 2\pi$ be a curve in $M \supset B$ that is not homotopic to zero. Let T_1 and T_2 be concentric tori (or Klein bottles) enclosing both C and N. Then a slide is the identity outside T_2 and inside T_1, while between T_1 and T_2 points move along circles "parallel" to the tori by an angle that increases from 0 to 2π as one moves inward from T_2 to T_1.

Table 2
Symmetry groups G for manifolds $M = \mathbf{R}^3 \# N$, N prime.

N	G	$R_{2\pi} \approx 1$? a)
$L(p,q)$	same as $\pi_0(D_N)$ of table 1	yes
octahedral space	O^*	no
truncated cube space	O^*	no
dodecahedral space	I^*	no
prism manifolds S^3/D^*_m	O^*, $m = 2$	
	D^*_{2m}, $m \geqslant 4$, even	$R_{2\pi} \approx 1 \Leftrightarrow m$ is odd
	D_{2m}, m odd	
$S^3/T'_{8\cdot 3^k}$, $S^3/D'_{2^k(2n+1)}$	?	no, yes
$S^3/(H \times Z_p)$	?	same as for S^3/H
$S^1 \times S^2$, $S^1 \underset{\sim}{\times} S^2$, $S^1 \times P^2$	$Z_2 \times Z_2$	yes
$K(\Pi, 1)$	$G/Z_2 \approx \mathrm{Aut}^+(\Pi)$ if $K(\Pi, 1)$ is "sufficiently large"	no

a) When N is not prime, $R_{2\pi} \approx 1$ if and only if $R_{2\pi} \approx 1$ for each prime factor.

group $\mathcal{S}$ generated by slides commutes with the symmetry group of each prime factor. For example, if N_1 and N_2 are not handles, $\mathcal{S}$ for $M = \mathbf{R}^3 \# N_1 \# N_2$ is just the free group generated by slides of N_1 about loops of N_2 and vice-versa: $\mathcal{S} = \pi_1(N_2) * \pi_1(N_1) \approx \pi_1(M)$. (For p handles the group G is analyzed by Laudenbach [19].) The subgroup of G generated by interchanges is the direct product of the permutation groups for the sets of identical prime factors (permutations do not, however, commute with symmetry groups of individual factors or with slides).

We are indebted to J. Arnold, J. Rubinstein, R. Sorkin, and J. Wagner for helpful discussions and to Rubinstein and C. Hodgson for communicating their results to us prior to publication.

References

[1] J.L. Friedman and R.D. Sorkin, Phys. Rev. Lett. 44 (1980) 1100.
[2] J.L. Friedman and R.D. Sorkin, Gravity Research Foundation Essay (April 1981), Gen. Rel. Grav., to be published.
[3] C.J. Isham, Phys. Lett. 106B (1981) 188.
[4] C.J. Isham, in: Proc. 1981 Nuffield quantum gravity Workshop, to be published.
[5] R.D. Sorkin, to be published.
[6] H. Seifert and W. Threfall, A textbook of topology (Academic Press, New York, 1980) p. 213.
[7] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.
[8] C.G. Callan, R.F. Dashen and D.J. Gross, Phys. Lett. 63B (1976) 334.
[9] C. Isham, private communication.
[10] J. Hempel, 3-Manifolds, Ann. Math. Stud. 86 (Princeton, 1976) p. 171.
[11] R. Schoen and S.-T. Yau, Phys. Rev. Lett. 43 (1979) 1457.
[12] M. Gromov and H.B. Lawson, Ann. Math. 111 (1980) 209.
[13] A. Hatcher, Linearization in 3-dimensional topology, preprint, UCLA.
[14] J.H. Rubinstein, Trans. Am. Math. Soc. 251 (1979) 129; J.H. Rubinstein and J.S. Birman, Homeotopy groups of some non-Haken manifolds, preprint (1982).
[15] C. Hodgson and J.H. Rubinstein, Involutions and diffeomorphisms of lens spaces, to be published.
[16] P. Orlick, Seifert Manifolds, Lecture Notes in Mathematics, Vol. 291 (Springer, Berlin, 1972).
[17] W. Threfall and H. Seifert, Math. Ann. 104 (1930–31) 1.
[18] H.R. Gluck, Bull. AMS 67 (1961) 586.
[19] F. Laudenbach, Asterisque 12 (1974) 1.
[20] N. Steenrod, The Topology of Fiber Bundles (Princeton, 1974) p. 90.
[21] H. Hendricks, Bull. Soc. Math. France, Memoire 53 (1977) 81, § 4.3.
[22] C. de Sâ and C. Rourke, Bull. AMS 1 (1979) 1.