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The Isometry Groups of Asymptotically Flat, Asymptotically Empty Space–Times with Timelike ADM Four–Momentum

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Abstract: We give a complete classification of all connected isometry groups, together with their actions in the asymptotic region, in asymptotically flat, asymptotically vacuum space–times with timelike ADM four–momentum.

1. Introduction

In any physical theory a privileged role is played by those solutions of the dynamical equations which exhibit symmetry properties. For example, according to a current paradigm, there should exist a large class of isolated gravitating systems which are expected to settle down towards a stationary state, asymptotically in time, outside of black hole regions. If that is the case, a classification of all such stationary states would give exhaustive information about the large–time dynamical behavior of the solutions under consideration. More generally, one would like to understand the global structure of all appropriately regular space–times exhibiting symmetries. Now the local structure of space–times with Killing vectors is essentially understood, the reader is referred to the book [20], a significant part of which is devoted to that question. However, in that reference, as well as in most works devoted to those problems, the global issues arising in this context are not taken into account. In this paper we wish to address the question, what is the structure of the connected component of the identity of the group of isometries of space–times which are asymptotically flat in space–like directions, when the condition of time–likeness of the ADM four–momentum p^μ is imposed? Recall that the time–likeness of p^μ can be established when the Einstein tensor satisfies a positivity condition, and when the space–time contains an appropriately regular spacelike surface, see [4] for a recent discussion and a list of references. Thus the condition of time–likeness of p^μ is a rather weak form of imposing global restrictions on the space–time

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under consideration. The reader should note that we do not require p^0 to be positive, so that our results also apply to space–times with negative mass, as long as the total four–momentum is time–like.

In asymptotically flat space–times one expects Killing vectors to "asymptotically look like" their counterparts in Minkowski space–time – in [4, Proposition 2.1] we have shown that *at the leading order* this is indeed the case (see also Proposition 2.1 below). This allows one to classify the Killing vectors into "boosts", "translations", etc., according to their leading asymptotic behavior. There exists a large literature concerning the case in which one of the Killing vectors is a time–like translation – *e.g.*, the theory of uniqueness of black holes – but no exhaustive analysis of what Killing vectors are kinematically allowed has been done so far. This might be due to the fact that for Killing vector fields with a rotation–type leading order behaviour, the next to leading order terms are essential to analyse the structure of the orbits, and it seems difficult to control those without some overly restrictive hypotheses on the asymptotic behaviour of the metric. In this work we overcome this difficulty, and prove the following (the reader is referred to Sect. 2 for the definition of a boost–type domain, and for a detailed presentation of the asymptotic conditions used in this paper):

Theorem 1.1. *Let $(M, g_{\mu\nu})$ be a space–time containing an asymptotically flat boost–type domain Ω, with time–like (non–vanishing) ADM four momentum p^μ, with fall–off exponent $1/2 < \alpha < 1$ and differentiability index $k \geq 3$ (see Eq. (2.2) below). We shall also assume that the hypersurface $\{t = 0\} \subset \Omega$ can be Lorentz transformed to a hypersurface in Ω which is asymptotically orthogonal to p^μ. Suppose moreover that the Einstein tensor $G_{\mu\nu}$ of $g_{\mu\nu}$ satisfies in Ω the fall–off condition*

$$G_{\mu\nu} = \mathcal{O}(r^{-3-\epsilon}), \qquad \epsilon > 0 . \tag{1.1}$$

Let X^μ be a non–trivial Killing vector field on Ω, let $\phi_s[X]$ denote its (perhaps only locally defined) flow. Replacing X^μ by an appropriately chosen multiple thereof if necessary, one has:

1. *There exists $R_1 \geq 0$ such that $\phi_s[X](p)$ is defined for all $p \in \Sigma_{R_1} \equiv \{(0, \vec{x}) \in \Omega : r(\vec{x}) \geq R_1\}$ and for all $s \in [0, 1]$.*
2. *There exists a constant $a \in \mathbf{R}$ such that, in local coordinates on Ω, for all $x^\mu = (0, \vec{x})$ as in point 1 we have*

$$\phi_1^\mu[X] = x^\mu + ap^\mu + \mathcal{O}_k(r^{-\alpha}) .$$

3. *If $a = 0$, then $\phi_1[X](p) = p$ for all p for which $\phi_1[X](p)$ is defined.*

The reader should notice that Theorem 1.1 excludes boost-type Killing vectors. This feature is specific to asymptotic flatness at spatial infinity, see [6] for a large class of vacuum space–times with boost symmetries which are asymptotically flat in light–like directions. The theorem is sharp, in the sense that the result is not true if p^μ is allowed to vanish or to be non–time–like.

When considering asymptotically flat space–times with more than one Killing vector, it is customary to assume that there exists a linear combination of Killing vectors the orbits of which are periodic (and has an axis — see below). However no justification of this property of Killing orbits has been given so far, except perhaps in some special situations. Theorem 1.1 allows us to show that this is necessarily the case. While this property, appropriately understood, can be established without making the hypothesis of completeness of the orbits of the Killing vector fields, the statements become somewhat awkward. For the sake of simplicity let us therefore assume that we have an action of a connected non–trivial group G_0 on $(M, g_{\mu\nu})$ by isometries. Using Theorem 1.1 together with the results of [4] we can classify all the groups and actions. Before doing that we need to introduce some terminology. Consider a space–time $(M, g_{\mu\nu})$ with a Killing vector field X. Then $(M, g_{\mu\nu})$ will be said to be:

1. *Stationary*, if there exists an asymptotically Minkowskian coordinate system $\{y^\mu\}$ on (perhaps a subset of) Ω, with y^0 — a time coordinate, in which $X = \partial/\partial y^0$. When the orbits of X are complete we shall require that they are diffeomorphic to **R**, and that $\Sigma_R \equiv \{t = 0, r(\vec{x}) \geq R\}$ intersects the orbits of X only once, at least for R large enough.
2. *Axisymmetric*, if X^μ has complete periodic orbits. Moreover X^μ will be required to have an axis, that is, the set $\{p : X^\mu(p) = 0\} \neq \emptyset$.
3. *Stationary-rotating* (compare [14]), if the matrix $\sigma^\mu_\nu = \lim_{r\to\infty} \partial_\nu X^\mu$ is a rotation matrix, that is, σ^μ_ν has a timelike eigenvector a^μ, with zero eigenvalue[1]. Let $\phi_t[X]$ denote the flow of X. We shall moreover require that there exists $T > 0$ such that $\phi_T[X](p) \in I^+(p)$ for p in the exterior asymptotically flat 3-region Σ_{ext}.
4. *Stationary–axisymmetric*, if there exist on M two *commuting* Killing vector fields X_a, $a = 1, 2$, such that $(M, g_{\mu\nu})$ is stationary with respect to X_1 and axisymmetric with respect to X_2,
5. *Spherically symmetric*, if, in an appropriate coordinate system on Ω, $SO(3)$ acts on M by rotations of the spheres r = const, t = const′ in Ω, at least for $t = 0$ and r large enough.
6. *Stationary–spherically symmetric*, if $(M, g_{\mu\nu})$ is stationary and spherically symmetric.

We have the following:

Theorem 1.2. *Under the conditions of Theorem 1.1, let G_0 denote the connected component of the group of all isometries of $(M, g_{\mu\nu})$. If G_0 is non–trivial, then one of the following holds:*

1. *G_0 = **R**, and $(M, g_{\mu\nu})$ is either stationary, or stationary–rotating.*
2. *$G_0 = U(1)$, and $(M, g_{\mu\nu})$ is axisymmetric.*
3. *G_0 = **R** $\times U(1)$, and $(M, g_{\mu\nu})$ is stationary–axisymmetric.*
4. *$G_0 = SO(3)$, and $(M, g_{\mu\nu})$ is spherically symmetric.*
5. *G_0 = **R** $\times SO(3)$, and $(M, g_{\mu\nu})$ is stationary–spherically symmetric.*

We believe that the condition that Ω be a boost–type domain is unnecessary. Recall, however, that this condition is reasonable for vacuum space–times [9], and one expects it to be reasonable for a large class of couplings of matter fields to gravitation, including electro–vacuum space–times. We wish to point out that in our proof that condition is needed to exclude boost–type Killing vectors, in Proposition 2.2 below, as well as to exclude causality violations in the asymptotic region. We expect that it should be possible to exclude the boost–type Killing vectors purely by an initial data analysis, using the methods of [4]. If that turns out to be the case, the only “largeness requirements” left on $(M, g_{\mu\nu})$ would be the much weaker conditions[2] needed in Proposition 2.3 below.

[1] If σ^μ_ν has a timelike eigenvector a^μ, we can find a Lorentz frame so that $a^\mu = (a, 0, 0, 0)$. In that frame σ^μ_ν satisfies $\sigma^0_\nu = \sigma^\mu_0 = 0$, so that it generates space–rotations, if non–vanishing.

[2] Those global considerations of the proof of Theorem 1.2 which use the structure of Ω can be carried through under the condition (2.15), provided that the constants C_1 and $\hat{C}_1$ appearing there are replaced by some appropriate larger constants. The reader should also note that these considerations are unnecessary when Σ_R is assumed to be achronal.

Let us also mention that in stationary space–times with more than one Killing vector all the results below can be proved directly by an analysis of initial data sets, so that no "largeness" conditions on $(M, g_{\mu\nu})$ need to be imposed — see [3].

Let us finally mention that the results here settle in the positive Conjecture 3.2 of [13], when the supplementary hypothesis of existence of at least two Killing vectors is made there.

We find it likely that there exist no electro–vacuum, asymptotically flat space–times which have no black hole region, which are stationary–rotating and for which $G_0 = \mathbf{R}$. A similar statement should be true for domains of outer communications of regular black hole space–times. It would be of interest to prove this result. Let us also point out that the Jacobi ellipsoids [7] provide a Newtonian example of solutions with a one dimensional group of symmetries with a "stationary–rotating" behavior.

2. Definitions, Proofs

Let W be a vector field, throughout we shall use the notation $\phi_t[W]$ to denote the (perhaps defined only locally) flow generated by W.

Consider a subset Ω of $\mathbf{R}^4$ of the form

$$\Omega = \{(t, \vec{x}) \in \mathbf{R} \times \mathbf{R}^3 : r((t, \vec{x})) \geq R, |t| \leq f(r(\vec{x}))\}, \tag{2.1}$$

for some constant $R \geq 0$ and some function $f(r) \geq 0, f \not\equiv 0$. We shall consider only *non–decreasing* functions f. Here and elsewhere, by a slight abuse of notation, we write

$$r((t, \vec{x})) = r(\vec{x}) = \sqrt{\sum_{i=1}^{3}(x^i)^2}\,.$$

Let α be a positive constant; Ω will be called *a boost–type domain* if $f(r) = \theta r + C$ for some constants $\theta > 0$ and $C \in \mathbf{R}$ (*cf.* also [9]).

Let ϕ be a function defined on Ω. For $\beta \in \mathbf{R}$ we shall say that $\phi = \mathcal{O}_k(r^\beta)$ if $\phi \in C^k(\Omega)$, and if there exists a function $C(t)$ such that we have

$$0 \leq i \leq k \qquad |\partial_{\alpha_1} \cdots \partial_{\alpha_i} \phi| \leq C(t)(1+r)^{\beta - i}\,.$$

We write $\mathcal{O}(r^\beta)$ for $\mathcal{O}_0(r^\beta)$. We say that $\phi = o(r^\beta)$ if $\lim_{r\to\infty, t=\text{const}} r^{-\beta}\phi(t, x) = 0$. A metric on Ω will be said to be asymptotically flat if there exist $\alpha > 0$ and $k \in \mathbf{N}$ such that

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_k(r^{-\alpha})\,, \tag{2.2}$$

and if there exists a function $C(t)$ such that

$$|g_{\mu\nu}| + |g^{\mu\nu}| \leq C(t)\,, \tag{2.3}$$

$$g_{00} \leq -C(t)^{-1}, \qquad g^{00} \leq -C(t)^{-1}\,, \tag{2.4}$$

$$\forall X^i \in \mathbf{R}^3 \quad g_{ij}X^iX^j \geq C(t)^{-1}\sum (X^i)^2\,. \tag{2.5}$$

Here and throughout $\eta_{\mu\nu}$ is the Minkowski metric.

Given a set Ω of the form (2.1) with a metric satisfying (2.2)–(2.5), to every slice $\{t = \text{const}\} \subset \Omega$ one can associate in a unique way the ADM four–momentum vector p^μ (see [10, 2]), provided that $k \geq 1$, $\alpha > 1/2$, and that the Einstein tensor satisfies the fall–off condition (1.1). Those conditions also guarantee that p^μ will not depend upon which hypersurface t = const has been chosen. The ADM four–momentum of Ω will be defined as the four–momentum of any of the hypersurface $\{t = \text{const}\} \subset \Omega$.

We note the following useful result:

Proposition 2.1. *Consider a metric $g_{\mu\nu}$ defined on a set Ω as in (2.1) (with a non–decreasing function f), and suppose that $g_{\mu\nu}$ satisfies (2.2)–(2.5) with $k \geq 2$ and $0 < \alpha < 1$. Let X^μ be a Killing vector field defined on Ω. Then there exist numbers $\sigma_{\mu\nu} = \sigma_{[\mu\nu]}$ such that*

$$X^\mu - \sigma^\mu{}_\nu x^\nu = \mathcal{O}_k(r^{1-\alpha}) \, , \tag{2.6}$$

with $\sigma^\mu{}_\nu \equiv \eta^{\mu\alpha}\sigma_{\alpha\nu}$. If $\sigma_{\mu\nu} = 0$, then there exist numbers A^μ such that

$$X^\mu - A^\mu = \mathcal{O}_k(r^{-\alpha}) \, . \tag{2.7}$$

If $\sigma_{\mu\nu} = A^\mu = 0$, then $X^\mu \equiv 0$.

Proof. The result follows from Proposition 2.1 of [4], applied to the slices $\{t = \text{const}\}$, except for the estimates on those partial derivatives of X in which $\partial/\partial t$ factors occur. Those estimates can be obtained from the estimates for the space–derivatives of Proposition 2.1 of [4] and from the equations

$$\nabla_\mu \nabla_\nu X_\alpha = R^\lambda{}_{\mu\nu\alpha} X_\lambda \, , \tag{2.8}$$

which are a well known consequence of the Killing equations. $\square$

The proofs of Theorems 1.1 and 1.2 require several steps. Let us start by showing that boost–type Killing vectors are possible only if the ADM four–momentum is spacelike or vanishes:

Proposition 2.2. *Let $g_{\mu\nu}$ be a twice differentiable metric on a boost–type domain Ω, satisfying (2.2)–(2.5), with $\alpha > 1/2$ and with $k \geq 2$. Suppose that the Einstein tensor $G_{\mu\nu}$ of $g_{\mu\nu}$ satisfies*

$$G_{\mu\nu} = \mathcal{O}(r^{-3-\epsilon}), \qquad \epsilon > 0 \, .$$

Let X^μ be a Killing vector field on Ω, set

$$\sigma^\mu{}_\nu \equiv \lim_{r\to\infty} \frac{\partial X^\mu}{\partial x^\nu} \tag{2.9}$$

(those limits exist by Proposition 2.1). Then the ADM four–momentum p^μ of Ω satisfies

$$\sigma_\mu{}^\nu p^\mu = 0 \, . \tag{2.10}$$

Proof. If $\sigma^\mu{}_\alpha = 0$ there is nothing to prove, suppose thus that $\sigma^\mu{}_\alpha \not\equiv 0$. Let $\Omega^\mu{}_\nu$ be a solution of the equation

$$\frac{d\Omega^\mu{}_\nu}{ds} = \sigma^\mu{}_\alpha \Omega^\alpha{}_\nu \, .$$

It follows from Proposition 2.1 that the flow $\phi_t[X](p)$ is defined for all $t \in [-\alpha, \alpha]$ and for all $p \in \Sigma_{R_1} \equiv \{t = 0, r(p) \geq R_1\} \subset \Omega$ for some constants α and R_1. By [11, Theorem 1], in local coordinates we have

$$\begin{aligned} \phi_t^\mu[X] &= \Omega^\mu{}_\nu(t) x^\nu + \mathcal{O}_k(r^{1-\alpha}) \, , \\ \tfrac{\partial \phi_t^\mu[X]}{\partial x^\nu} &= \Omega^\mu{}_\nu(t) + \mathcal{O}_{k-1}(r^{1-\alpha}) \, . \end{aligned}$$

The error terms above satisfy appropriate decay conditions so that the ADM four–momentum

$$p_\mu(\phi_t[X](\Sigma_{R_1})) = \int_{\phi_t[X](\Sigma_{R_1})} U_\mu^{\alpha\beta} dS_{\alpha\beta}$$

is finite and well–defined. Here $dS_{\mu\nu} = \iota_{\partial_\mu}\iota_{\partial_\nu} dx^0 \wedge \ldots \wedge dx^3$, ι_X denotes the inner product of a vector X with a form, and (*cf.*, *e.g.*, [11])

$$U_\mu^{\alpha\beta} = \delta_\lambda^{[\alpha}\delta_\nu^\beta\delta_\mu^{\gamma]}\eta^{\lambda\rho}\eta_{\gamma\sigma}\partial_\rho g^{\nu\sigma} \ .$$

As is well known (see [11] for a proof under the current asymptotic conditions, *cf.* also [5, 1]), under boosts the ADM four–momentum transforms like a four–vector, that is,

$$p^\mu(\phi_t[X](\Sigma_{R_1})) = \Omega^\mu{}_\nu(t)p^\nu(\Sigma_{R_1}) \ . \tag{2.11}$$

On the other hand, the $\phi_t^\mu[X]$'s are isometries, so that

$$g_{\alpha\beta}(\phi_t^\mu[X](x))\frac{\partial\phi_t^\alpha[X]}{\partial x^\mu}(x)\frac{\partial\phi_t^\beta[X]}{\partial x^\nu}(x) = g_{\mu\nu}(x) \ ,$$

which gives

$$U_\alpha^{\mu\nu}(\phi_t^\mu[X](x))\Omega^\sigma{}_\mu(t)\Omega^\rho{}_\nu(t) = \Omega^\gamma{}_\alpha(t)U_\gamma^{\rho\sigma}(x) + \mathcal{O}(r^{-1-2\alpha}) \ . \tag{2.12}$$

Equations (2.11) and (2.12) give, for all t,

$$\Omega^\sigma{}_\mu(t)p_\sigma = p_\mu \ , \tag{2.13}$$

and (2.10) follows by t–differentiation of Eq. (2.13). $\square$

Suppose, now, that the ADM four–momentum p^μ of the hypersurface $\{t = 0\}$ is timelike. If Ω is large enough we can find a boost transformation Λ such that the hypersurface $\Lambda(\{t = 0\})$ is asymptotically orthogonal to p^μ. It then follows by Proposition 2.2 that the matrix σ defined in Eq. (2.9) has vanishing 0-components in that Lorentz frame, and therefore generates space rotations. We need to understand the structure of orbits of such Killing vectors. This is analysed in the proposition that follows:

Proposition 2.3. *Let $g_{\mu\nu}$ be a metric on a set Ω as in Eq. (2.1), and suppose that $g_{\mu\nu}$ satisfies the fall-off condition (2.2) with $0 < \alpha < 1$ and $k \geq 2$. Let X^μ be a Killing vector field defined on Ω, and suppose that*

$$Z^\mu\partial_\mu \equiv X^\mu\partial_\mu - \omega^i{}_j x^j\partial_i = o(r) \ , \qquad \partial_\sigma Z^\mu = o(1) \ , \tag{2.14}$$

with $\omega^i{}_j$ — a (non–trivial) antisymmetric matrix with constant coefficients, normalized such that $\omega^i{}_j\omega^j{}_i = -2(2\pi)^2$. (It follows from Proposition 2.1 that there exist constants $C_1, \hat{C}_1$ such that $|X^0| \leq C_1 r^{1-\alpha} + \hat{C}_1$ on $\{t = 0\} \subset \Omega$.) Suppose that the function f in (2.1) satisfies

$$f(r) \geq C_2 r^{1-\alpha} + \hat{C}_1 \ , \tag{2.15}$$

where C_2 is any constant larger than C_1. Let ϕ_s denote the flow of X^μ. Then:

1. *There exists $R_1 \geq R$ such that $\phi_s(p)$ is well defined for $p \in \Sigma_{R_1} \equiv \{t = 0, r \geq R_1\} \subset \Omega$ and for $s \in [0, 1]$. For those values of s we have $\phi_s(\Sigma_{R_1}) \subset \Omega$.*

2. *There exist constants A^μ such that, in local coordinates on Ω, for all $x^\mu \in \Sigma_{R_1}$ we have*
$$\phi_1^\mu = x^\mu + A^\mu + \mathcal{O}_{k-1}(r^{-\alpha}) \ . \tag{2.16}$$
3. *If $A^\mu = 0$, then $\phi_1(p) = p$ for all p for which $\phi_1(p)$ is defined.*

Remark. The hypothesis that $\lim_{r\to\infty} \partial_i X^0 = 0$, which is made in (2.14), is not needed for points 2 and 3 above to hold, provided one assumes that the conclusions of point 1 hold.

Proof. Point 1 follows immediately from the asymptotic estimates of Proposition 2.1 and the defining equations for ϕ_s^μ,
$$\frac{d\phi_s^\mu}{ds} = X^\mu \circ \phi_s^\mu .$$
To prove point 2, let $R^i{}_j(s)$ be the solution of the equation
$$\frac{dR^i{}_j}{ds} = \omega^i{}_k R^k{}_j \ ,$$
with initial condition $R^i{}_j(0) = \delta^i{}_j$, set $R^0{}_0(s) = 1$, $R^0{}_i(s) = 0$. We have the variation–of–constants formula
$$\phi_s^\mu(x) = R^\mu_\nu(s) x^\nu + \int_0^s R^\mu{}_\nu(s-t) Z^\nu(\phi_t(x)) dt,$$
from which we obtain, in view of Proposition 2.1,
$$\tfrac{\partial\phi_1^\mu}{\partial x^\nu} - \delta^\mu{}_\nu = \mathcal{O}_{k-1}(r^{-\alpha}), \tag{2.17}$$
$$\phi_1^\mu - x^\mu = \mathcal{O}_k(r^{1-\alpha}). \tag{2.18}$$
Set $y^\mu(x) = \phi_1^\mu(x)$. As $y^\mu(x^\nu)$ is an isometry, we have the equations
$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\sigma_{\mu\nu}(x) \frac{\partial y^\alpha}{\partial x^\sigma} - \Gamma^\alpha_{\beta\gamma}(y(x)) \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\gamma}{\partial x^\nu}. \tag{2.19}$$
From (2.17)–(2.18) we obtain
$$\begin{aligned} \frac{\partial^2 (y^\alpha - x^\alpha)}{\partial x^\mu \partial x^\nu} &= \Gamma^\alpha_{\mu\nu}(x) - \Gamma^\alpha_{\mu\nu}(y(x)) + \mathcal{O}_{k-1}(r^{-1-2\alpha}) \\ &= (y^\rho(x) - x^\rho) \int_0^1 \partial_\rho \Gamma^\alpha_{\mu\nu}(tx + (1-t)y(x)) dt + \mathcal{O}_{k-1}(r^{-1-2\alpha}) \\ &= \mathcal{O}_{k-2}(r^{-1-2\alpha}) \ . \end{aligned} \tag{2.20}$$
We can integrate this inequality in r to obtain
$$\frac{\partial (y^\alpha - x^\alpha)}{\partial x^\mu} = \mathcal{O}_{k-1}(r^{-2\alpha}) \ .$$
If $2\alpha > 1$, the Lemma the Appendix A of [11] shows that the limits $\lim_{r\to\infty, t=0}(y^\alpha - x^\alpha) = A^\alpha$ exist and we get
$$y^\alpha - x^\alpha = A^\alpha + \mathcal{O}_k(r^{1-2\alpha}) \ .$$

Otherwise, decreasing α slightly if necessary, we may assume that $2\alpha < 1$, in which case we simply obtain

$$y^\alpha - x^\alpha = \mathcal{O}_k(r^{1-2\alpha}) \ .$$

If the last case occurs we can repeat this argument $\ell - 1$ times to obtain $\mathcal{O}(r^{-1-(\ell+1)\alpha})$ at the right–hand–side of (2.20) until $-1-(\ell+1)\alpha < -2$; at the last iteration we shall thus obtain $\mathcal{O}(r^{-2-\epsilon})$ there, with some $\epsilon > 0$. We can again use the Lemma of the Appendix A of [11] to conclude that the limits $\lim_{r\to\infty, t=0}(y^\alpha - x^\alpha) = A^\alpha$ exist. An iterative argument similar to the one above applied to (2.20) gives then

$$\xi^\alpha \equiv y^\alpha - x^\alpha - A^\alpha = \mathcal{O}_k(r^{-\alpha}) \ , \tag{2.21}$$

which establishes point 2.

Suppose finally that A^μ vanishes. Equation (2.19) implies an inequality of the form

$$\left| \frac{\partial^2 (y^\alpha - x^\alpha)}{\partial x^\mu \partial x^\nu} \right| \le C(|\partial\Gamma||y-x| + |\Gamma||\partial(y-x)|), \tag{2.22}$$

for some constant C. A standard bootstrap argument using (2.22), (2.17) and (2.18) shows that for all $\sigma \ge 0$ we have

$$\lim_{r\to\infty} [r^\sigma |y-x| + r^\sigma |\partial(y-x)|] = 0. \tag{2.23}$$

Define

$$F = r^{\beta-2}|y-x|^2 + r^\beta |\partial(y-x)|^2. \tag{2.24}$$

Choosing β large enough one finds from (2.22) that

$$\frac{\partial F}{\partial r} \ge 0. \tag{2.25}$$

This implies

$$R_2 \le r \le r_1 \Rightarrow F(r_1) \ge F(r) \ge 0. \tag{2.26}$$

Passing with $r_1 \to \infty$ from (2.23) we obtain $\phi_1(x) = x$ for $x \in \Sigma_{R_1}$. ϕ_1 is therefore an isometry which reduces to an identity on a spacelike hypersurface, and point 3 follows from [12, Lemma 2.1.1]. □

We are ready now to pass to the proof of Theorem 1.1:

Proof of Theorem 1.1. Let $y^\alpha(x^\beta)$ be defined as in the proof of Proposition 2.3, as it is an isometry we have the equation:

$$g_{\mu\nu}(y(x)) \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} = g_{\alpha\beta}(x) \ . \tag{2.27}$$

Set $\xi_\alpha = \eta_{\alpha\beta}\xi^\beta$, where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$, with ξ defined by eq. (2.21). Equations (2.21) and (2.27) together with the asymptotic form of the metric, Eq. (2.2), give

$$\frac{\partial \xi_\alpha}{\partial x^\beta} + \frac{\partial \xi_\beta}{\partial x^\alpha} + g_{\alpha\beta}(x^\sigma + A^\sigma + \xi^\sigma) - g_{\alpha\beta}(x^\sigma) = \mathcal{O}_{k-1}(r^{-1-2\alpha}) \ . \tag{2.28}$$

Suppose first that $A^\sigma \not\equiv 0$; we have

$$
\begin{aligned}
&g_{\alpha\beta}(x^\sigma + A^\sigma + \xi^\sigma) - g_{\alpha\beta}(x^\sigma) \\
&\quad = \frac{\partial g_{\alpha\beta}}{\partial x^\rho}(x^\sigma)A^\rho + \int_0^1 \left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho}(x^\sigma + s(A^\sigma + \xi^\sigma))(A^\rho + \xi^\rho)) - \frac{\partial g_{\alpha\beta}}{\partial x^\rho}(x^\sigma)A^\rho \right) ds \\
&\quad = \frac{\partial g_{\alpha\beta}}{\partial x^\rho}(x^\sigma)A^\rho + \mathcal{O}(r^{-1-2\alpha}) \ .
\end{aligned}
$$

A similar calculation for the derivatives of $g_{\alpha\beta}$ gives

$$
g_{\alpha\beta}(x^\sigma + A^\sigma + \xi^\sigma) - g_{\alpha\beta}(x^\sigma) = \frac{\partial g_{\alpha\beta}}{\partial x^\rho}(x^\sigma)A^\rho + \mathcal{O}_{k-2}(r^{-1-2\alpha}) \ . \tag{2.29}
$$

In a neighbourhood of Σ_{R_1} define a vector field Y^μ by

$$
Y^\mu = \xi^\mu + A^\mu \ .
$$

It follows from (2.28)–(2.29) that Y^μ satisfies the equation

$$
\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = \mathcal{O}_{k-2}(r^{-1-2\alpha}) \ .
$$

By hypothesis we have $k \geq 3$ and $2\alpha > 1$, we can thus use [4, Proposition 3.1] to conclude that A^μ must be proportional to p^μ. The remaining claims follow directly by Proposition 2.3. $\Box$

To prove Theorem 1.2 we shall need two auxiliary results:

Proposition 2.4. *Under the hypotheses of Prop. 2.1, let W be a non–trivial Killing vector field defined on Ω. Suppose that there exists R_1 such that for $p \in \Sigma_{R_1}$ the orbits $\phi_s[W](p)$ are defined for $s \in [0,1]$, with $\phi_1[W](p) = p$. Assume moreover that there exists a non–vanishing antisymmetric matrix with constant coefficients $\omega^i{}_j$ such that $W^\mu\partial_\mu - \omega^i{}_j x^j \partial_i = o(r)$. Then the set $\{p : W(p) = 0\}$ is not empty.*

Remark. The following half–converse to Proposition 2.4 is well known: Let W be a Killing vector field on a Lorentzian manifold M and suppose that $W(p) = 0$. If there exists a neighborhood $\mathcal{O}$ of p such that W is nowhere time–like on $\mathcal{O}$, then there exists $T > 0$ such that all orbits which are defined for $t \geq T$ are periodic.

Proof. Let ϕ_s denote the flow of W on Ω, and for $p \in \Sigma_{R_1}$ define

$$
\bar{t}(p) = \int_0^1 t \circ \phi_s(p) ds, \tag{2.30}
$$

$$
\bar{r}(p) = \int_0^1 r \circ \phi_s(p) ds. \tag{2.31}
$$

Note that $(\phi_s)_*$ asymptotes to the matrix $R^\mu{}_\nu(s)$ defined in the proof of Prop. 2.3, which gives

$$
\nabla \bar{r} = \int_0^1 (\phi_s)_*(\nabla r) \circ \phi_s(p) ds \approx \nabla r + \mathcal{O}(r^{-\alpha}).
$$

Similarly

$$
\nabla \bar{t} \approx \nabla t + \mathcal{O}(r^{-\alpha}).
$$

This shows that for R large enough the sets $S_{R,T} = \{p : \bar{r}(p) = R,\ \bar{t}(p) = T\}$ are differentiable spheres. Moreover

$$\bar{r} \circ \phi_s = \bar{r}, \qquad \bar{t} \circ \phi_s = \bar{t}, \tag{2.32}$$

so that W is tangent to $S_{R,T}$. As every continuous vector field tangent to a two–dimensional sphere has fixed points, the result follows. □

Proof of Theorem 1.2. Let $\mathfrak{g}$ denote the Lie algebra of G_0. As is well known [19, Vol. I, Chapitre VI, Theorem 3.4], to any element h of $\mathfrak{g}$ there is associated a unique Killing vector field $X^\mu(h)$, the orbit of which is complete.

Suppose first that $\mathfrak{g}$ is 1–dimensional. If the constant a of Theorem 1.1 vanishes, $(M, g_{\mu\nu})$ is axisymmetric by part 3 of Theorem 1.1 and by Proposition 2.4. If a does not vanish there are two cases to analyse. Consider first the case in which $\partial_\mu X^\nu \not\to 0$ as $r \to \infty$. Let us perform a Lorentz transformation so that the new hypersurface $t = 0$, still denoted by Σ_R, is asymptotically normal to p^μ. By Proposition 2.2 we must have $\lim_{r\to\infty} \partial_i X^0 = \lim_{r\to\infty} \partial_0 X^i = 0$, hence Proposition 2.3 applies. As M contains a boost–type domain for any T we can choose $p \in \Sigma_{R_1}$, with $r(p)$ large enough, so that $\phi_s[X](p)$ is defined for all $s \in [0, T]$, with $\phi_s[X](p) \neq p$ by (2.16). This shows that G_0 cannot be $U(1)$, hence $G_0 = \mathbf{R}$, and $(M, g_{\mu\nu})$ is stationary–rotating as claimed. The second case to consider is, by Proposition 2.1, that in which $X^\mu \to ap^\mu = A^\mu$ as $r \to \infty$ in Ω. We want to show that Σ_R is a global cross–section for $\phi_s[X]$, at least for R large enough. To do that, note that timelikeness of A^μ implies that we can choose R_2 large enough so that X^μ is transverse to Σ_{R_2}. Let (g_{ij}, K_{ij}) be the induced metric and the extrinsic curvature of Σ_{R_2}, and let $(\hat{M}, \hat{g}_{\mu\nu})$ be the Killing development of $(\Sigma_{R_2}, g_{ij}, K_{ij})$ constructed using the Killing vector field X^μ, see Sect. 2 of [4] for details. Define $\Psi : \hat{M} \to M_{R_2} \equiv \cup_{t\in\mathbf{R}} \phi_t[X](\Sigma_{R_2})$ by $\Psi(t, \vec{x}) = \phi_t[X](0, \vec{x})$. Then Ψ is a local isometry between $\hat{M}$ and M_{R_2}. Ψ is surjective by construction, and there exists a boost–type domain $\hat{\Omega}$ in $\hat{M}$ such that $\Psi|_{\hat{\Omega}}$ is a diffeomorphism between $\hat{\Omega}$ and Ω.

Suppose that Ψ is not injective, let us first show that this is equivalent to the statement that $\Psi^{-1}(\Sigma_{R_2})$ is not connected. Indeed, let $p = (t, \vec{x})$ and $q = (\tau, \vec{y})$ be such that $\Psi(p) = \Psi(q)$, then $\phi_{-t}(\Psi(p)) = \phi_{-t}(\Psi(q))$ so that $\Psi((0, \vec{x})) = \Psi((\tau - t, \vec{y}))$, which leads to $(\tau - t, \vec{y}) \in \Psi^{-1}(\Sigma_{R_2})$.

Consider any connected component $\hat{\Sigma}$ of $\Psi^{-1}(\Sigma_{R_2})$, as Ψ is a local isometry $\hat{\Sigma}$ is an asymptotically flat hypersurface in $\hat{M}$. By [11, Lemma 1 and Theorem 1], we have

$$\hat{\Sigma} = \{t = h(\vec{x}), \quad \vec{x} \in \mathcal{U} \in \mathbf{R}^3\} ,$$

where $\mathcal{U}$ contains $\mathbf{R}^3 \setminus B(R_3)$ for some $R_3 \geq R_2$. Morever there exists a Lorentz matrix $\Lambda^\mu{}_\nu$ such that

$$h(\vec{x}) = \Lambda^0{}_i X^i + O(r^{1-\alpha}) .$$

Note that the unit normal to $\hat{\Sigma}$ approaches, as $r \to \infty$, the Killing vector X, hence

$$\Lambda^\mu{}_\nu X^\nu = X^\mu \quad \Rightarrow \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0 .$$

It follows that $h(\vec{x}) = O(r^{1-\alpha})$, so that $\Psi((h(\vec{x}), \vec{x})) \in \Omega$ for $r(\vec{x}) \geq R_4$ for some constant $R_4 \geq R_3$.

Consider a point $q \in \Sigma_{R_4}$, then there exists a point $(0, \vec{x})$ such that $\Psi(0, \vec{x}) = q$ and a point $(h(\vec{y}), \vec{y}) \in \hat{\Sigma}$ such that $\Psi(h(\vec{y}), \vec{y})) = q$. This, however, contradicts that fact that $\Psi|_{\hat{\Omega}}$ is a diffeomorphism between the boost-type domain $\hat{\Omega}$ and Ω. We conclude that ψ is injective. It follows that ψ is a bijection, which implies that all the orbits through $p \in \Sigma_{R_2}$ are diffeomorphic to $\mathbf{R}$, and that they intersect Σ_{R_2} only once.

Suppose next that $\mathfrak{g}$ is two–dimensional. Then there exist on M two linearly independent Killing vectors X^μ_a, $a = 1, 2$. Propositions 2.2 and 2.3 lead to the following three possibilites:

i) There exist constants B^μ_a, $a = 1, 2$ such that $X^\mu_a - B^\mu_a = o(1)$. By [4, Prop. 3.1] we have $B^\mu_a = a_a p^\mu$ for some constants a_a. It follows that there exist constants $(\alpha, \beta) \not= (0, 0)$ such that $\alpha X^\mu_1 + \beta X^\mu_2 = o(1)$. Proposition 2.1 implies that $\alpha X^\mu_1 + \beta X^\mu_2 = 0$, which contradicts the hypothesis dim $\mathfrak{g} = 2$, therefore this case cannot occur. ii) There exist constants B^μ and $\omega^i{}_j = -\omega^j{}_i$ such that

$$X^\mu_1 - B^\mu = o(1), \qquad X^\mu_2\partial_\mu - \omega^i{}_j x^i \partial_j = o(r)\ . \tag{2.33}$$

Consider the commutator $[X_1, X_2]$. The estimates on the derivatives of X^μ_a of Proposition 2.1 give $[X_1, X_2]^0 = o(1)$, $[X_1, X_2]^i = o(r)$, so that by Prop. 2.1 the commutator $[X_1, X_2]$ either vanishes, or asymptotes a constant vector with vanishing time–component, hence spacelike. The latter case cannot occur in view of [4, Prop. 3.1], hence $[X_1, X_2] = 0$. It follows that $\phi_t[X_2 + \alpha X_1] = \phi_t[X_2] \circ \phi_t[\alpha X_1]$. Let ap^μ be the vector given by Theorem 1.1 for the vector field X^μ_2. In local coordinates we obtain

$$\phi^\mu_1[X_2 + \alpha X_1] = x^\mu + ap^\mu + \alpha B^\mu + \mathcal{O}(r^{-\alpha})\ .$$

By [4, Prop. 3.1] we have $B^\mu \sim p^\mu$, so that we can choose α so that $\phi^\mu_1[X_2 + \alpha X_1] = x^\mu + \mathcal{O}(r^{-\alpha})$. By point 3 of Theorem 1.1 we obtain $\phi_1[X_2 + \alpha X_1](p) = p$, hence all orbits of $X^\mu_2 + \alpha X^\mu_1$ are periodic with period 1. As p^μ is time–like, the orbits of X^μ_1 must be time–like in the asymptotic region. As before, those orbits cannot be periodic because the coordinates on Ω cover a boost–type region, hence they must be diffeomorphic to **R**. As $[X_1, X_2] = 0$, we obtain that G_0 is the direct product $\mathbf{R} \times U(1)$.

iii) For dim $\mathfrak{g} = 2$ the last case left to consider is that when there exist non–zero constants ω^a_{ij}, $a = 1, 2$, such that $X^\mu_a\partial_\mu - \omega^a_{ij} x^i \partial_j = o(r)$. Suppose that the antisymmetric matrices ω^a_{ij} do not commute, then by well known properties of $so(3)$ the matrices ω^a_{ij} together with the matrix $\omega^1_{ij}\omega^2_{jk} - \omega^2_{ij}\omega^1_{jk}$ are linearly independent. It follows that $[X_1, X_2]$ is a Killing vector linearly independent of X_1 and X_2 near infinity, whence everywhere in Ω. It is well known that the orbits of $[X_1, X_2]$ are complete when those of X_1 and X_2 are [19, Vol. I, Chapitre VI, Theorem 3.4], which implies that G_0 is at least three–dimensional, which contradicts dim $\mathfrak{g} = 2$. If the matrices ω^a_{ij} commute they are linearly dependent. Thus there exist constants $(\alpha, \beta) \not= (0, 0)$ such that $\alpha X^\mu_1 + \beta X^\mu_2 = o(r)$. By Proposition 2.1 the Killing vector field $\alpha X^\mu_1 + \beta X^\mu_2$ is a translational Killing vector, and the case here is reduced to point ii) above.

Let us turn now to the case of a three dimensional Lie algebra $\mathfrak{g}$. An analysis similar to the above shows that this can only be the case if three Killing vector fields X^μ_i, $i = 1, 2, 3$, on M can be chosen so that $X^\mu_i\partial_\mu - \epsilon_{ijk} x^j \partial_k = o(r)$. Moreover we must have $[X_i, X_j] = \epsilon_{ijk} X_k$. Then $\mathfrak{g}$ is the Lie algebra of $SO(3)$, so that $G_0 = SO(3)$, or its covering group $Spin(3) = SU(2)$ [18, p. 117, Problem 7]. Integrating over the group as in the proof of Proposition 2.4 (the integral $\int_0^1$ in Eqs. (2.30) –(2.31) should be replaced by an integral over the group G_0 with respect to the Haar measure) one can pass to a new coordinate system, defined perhaps only on a subset of Ω, such that the spheres t = const, r = const′ are invariant under G_0. G_0 must be $SO(3)$, as $SO(3)$ is[3] the largest group acting effectively on S^2. The proof of point 5) is left to the reader. □

[3] This can be seen as follows: Any isometry is uniquely determined by its action at one point of the tangent bundle. Since $SO(3)$ acts transitively on TS^2, no larger groups can act effectively there.

3. Concluding Remarks

Theorem 1.1 leaves open the intriguing possibility of a space–time which has *only one* Killing vector which, roughly speaking, behaves as a spacelike rotation accompanied by a time–like translation. We conjecture that this is not possible when the Einstein tensor $G_{\mu\nu}$ falls–off at a sufficiently fast rate, when global regularity conditions are imposed and when positivity conditions on $G_{\mu\nu}$ are imposed.

One would like to go beyond the classification of groups given here, and consider the whole group of isometries G, not only the connected component of the identity thereof G_0. Recall, *e.g.*, that a discrete group of conformal isometries acts on the critical space–times which arise in the context of the Choptuik effect [8, 17]. Let us first consider the case of time–periodic space–times. Clearly such space–times exist when no field equations or energy inequalities hold, so that the classification question becomes interesting only when some field equations or energy–inequalities are imposed. In the vacuum case some stationarity results have been obtained for spatially compact space–times by Galloway [15]. In the asymptotically flat context non–existence of periodic non–stationary vacuum solutions with an analytic Scri has been established by Papapetrou [21], *cf.* also Gibbons and Stewart [16]. The hypothesis of analyticity of Scri is, however, difficult to justify; moreover the example of boost–rotation symmetric space–times shows that the condition of asymptotic flatness in light–like directions might lead to essentially different behaviour, as compared to that which arises in the context of asymptotic flatness in space–like directions. One expects that non–stationary time–periodic vacuum space–times do not exist, but no satisfactory analysis of that possibility seems to have been done so far.

Another set of discrete isometries that might arise is that of discrete subgroups of the rotation group, time–reflections, space–reflections, etc. In those cases G/G_0 is compact. It is easy to construct initial data (g_{ij}, K_{ij}) on a compact or asymptotically flat manifold Σ which are invariant under a discrete isometry group, in such a way that the group H of all isometries of g_{ij} which preserve K_{ij} is *not* connected. By [12, Theorem 2.1.4] the group H will act by isometries on the maximal globally hyperbolic development $(M, g_{\mu\nu})$ of (Σ, g_{ij}, K_{ij}), and it is rather clear that in generic such situations the groups G of all isometries of $(M, g_{\mu\nu})$ will coincide with H. In this way one obtains space–times in which G/G_0 is compact. It is tempting to conjecture that for, say vacuum, globally hyperbolic space–times with a compact or asymptotically flat, appropriately regular, Cauchy surface, the quotient G/G_0 will be a finite set. The proof of such a result would imply non–existence of non–stationary time–periodic space–times, in this class of space–times.

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