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Particles in a stationary spherically symmetric gravitational field†

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Abstract. Dirac's equation has been considered in a Schwarzschild (Reissner–Nordstrøm) background. Resonances in a continuum of states are found, similar to the Klein–Gordon case. Also the solutions of the Dirac and Klein–Gordon equations are investigated for an extended gravitating source. Special attention has been given to the case in which the radius of the source tends towards its Schwarzschild radius. A limiting charge-to-mass ratio for black holes is obtained.

1. Introduction

One of the curiosities of Einstein's theory of gravitation is the appearance of event horizons and singularities. An horizon is characterized by a region in space–time, from which it is impossible to escape to infinity. The boundary of this region is given by a wavefront (null hypersurface) which just cannot escape to infinity. This event horizon generates a strong limitation on the causal relationships between different parts of space–time.

Singularities are positions in a space–time manifold at which the normal picture of space and time breaks down, caused e.g. by an infinite Riemannian curvature. It has been shown in a number of theorems that under very general conditions singularities and horizons appear in Einstein's theory (Misner *et al* 1973). This could possibly be used as an argument against Einstein's Lagrangian among competitive theories. On the other hand the x-ray source X1 in Cygnus may be, according to experimental data, a black hole (De Witt and De Witt 1972). Furthermore, a number of experiments performed just recently have proved many of Einstein's predictions up to an accuracy of about 3%.

So one is led to the question, whether the inclusion of matter fields and/or the dynamical treatment of the gravitational collapse by use of Einstein's Lagrangian leads to manifolds free from pathologies. To investigate this, we shall couple matter as a first quantized field to the (classical) gravitational field. This is an approximation, since the gravitational field is not quantized. It is expected, however, that this approximation is

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good enough to study global gravitational effects such as e.g. the macroscopic change of singularities like the classical Schwarzschild radius, under the influence of other fields, e.g. electron–positron, pion, nucleon–antinucleon fields. Therefore our programme can be characterized as an attempt to investigate the interaction of a geometry with a quantized matter field. The phenomenon guiding us in our research is the well known example of the supercritical electric fields ($Z = 137$ problem) which can be generated by superheavy nuclei and molecules (Pieper and Greiner 1969, Müller *et al* 1972). It has been shown for this case, that the normal neutral electron–positron vacuum breaks down for over-critical charges leading to a new stable charged vacuum†. The latter contains finite, quantized amounts of charge shielding, to some extent, the original central charge (Rafelski *et al* 1974). The charge of the vacuum is generated by the creation of e^+e^- pairs for which the electrons are bound. This is due to the charge asymmetry of the central Coulomb interaction. The question we would like to answer is whether similar effects happen in superstrong gravitational fields, which are—in contrast to the electromagnetic case—charge symmetric as long as there is no charge on the central mass.

Particle creation in non-static geometries (an expanding or collapsing universe for example) and in strong gravitational fields appearing in the vicinity of a black hole, has already been discussed. Among these works, Hawking's theory (Hawking 1974, 1975) of a 'thermal emission', in which he studies the influence of the time-dependent metric during a gravitational collapse upon a scalar field, is probably best known.

Ruffini and co-workers (Deruelle and Ruffini 1974, Damour *et al* 1976) found that pair creation in the region surrounding a Kerr–Newman black hole was possible, based on a static background geometry, in contrast to Hawking's theory. Pair creation occurs here as an example of 'Klein's paradox', in analogy to the previously mentioned situation of superstrong fields in quantum electrodynamics (QED) that have been discussed by Müller, Rafelski and Greiner during the last eight years (Pieper and Greiner 1969, Müller *et al* 1972, Rafelski *et al* 1974).

For the Klein–Gordon equation in a Kerr–Newman geometry Ruffini found a continuum of states with resonances in the energy region from $+mc^2$ to $-mc^2$. Based on classical arguments, those resonances were interpreted as quasi-bound states, which decay with finite probability towards the singularity, the probability of decay being given by the width of the resonance. Again we have a close analogy with the results of over-critical Coulomb fields where bound states have entered the negative energy continuum of the Dirac equation yielding quasi-bound resonance configurations. The width of those is a measure for the decay of the vacuum state.

However, it is not clear what the connection is between this continuum and the discrete energy levels of a scalar or bispinor field, which one would expect in the case of a weak gravitational field (without an event horizon). Therefore in the present work, we study the Dirac and Klein–Gordon equations in the field of an extended gravitating source (liquid drop) and investigate the structure of the energy spectrum for the source radius tending towards the Schwarzschild radius.

Furthermore, we present the solutions of Dirac's equation in a Schwarzschild and Reissner–Nordstrøm background and discuss the appearance of resonances in a 'pseudo'-continuum, as in the Klein–Gordon case. One of the interesting results will be the appearance of a limiting charge-to-mass density (charge-to-mass ratio) for black holes (see also Gibbons 1975).

† For reviews on the charged vacuum in over-critical fields see Müller (1976) and Greiner and Scheid (1976).

2. The Dirac equation in a static, spherically symmetric gravitational field

We consider Dirac's equation in a Schwarzschild (Reissner–Nordstrøm) field or more generally, in a static, spherically symmetric gravitational field (Brill and Wheeler 1957)

$$g_{ij} = \mathrm{diag}(\mathrm{e}^{\lambda}, r^2, r^2 \sin^2\theta, -\mathrm{e}^{\nu}) \tag{1}$$

where

$$\mathrm{e}^{\nu} = \mathrm{e}^{-\lambda} = 1 - \frac{2M_{\mathrm{G}}}{r} \qquad \text{(Schwarzschild)} \tag{2}$$

or

$$\mathrm{e}^{\nu} = \mathrm{e}^{-\lambda} = 1 - \frac{2M_{\mathrm{G}}}{r} + \frac{Q_{\mathrm{G}}^2}{r^2} \equiv \Omega^2 \qquad \text{(Reissner–Nordstrøm).} \tag{3}$$

In equations (2) and (3) we have introduced the mass and charge of the black hole, M_{G} and Q_{G} respectively, in terms of 'geometrized' units, which are connected to mass M and charge Q in ordinary units by $M_{\mathrm{G}} = GM/c^2$ and $Q_{\mathrm{G}}^2 = GQ^2/c^4$. In general, we shall characterize 'geometrized' units by a subscript G and set $\hbar = c = 1$.

The action principle for the Dirac field reads

$$\delta \int \mathrm{d}^4x \sqrt{-g}(\bar{\psi}\gamma^{\mu}\psi_{;\mu} - m\bar{\psi}\psi) = 0. \tag{4}$$

Here the semicolon denotes the covariant derivative of a bispinor. $\bar{\psi}$ denotes the conjugate of ψ, i.e.

$$\bar{\psi} = \mathrm{i}\psi^{+}\tilde{\gamma}^4. \tag{5}$$

Equation (4) leads to the covariant Dirac equation and its conjugate, respectively:

$$\gamma^k\psi_{;k} - m\psi = 0 \tag{6a}$$

$$\bar{\psi}_{;k}\gamma^k + m\bar{\psi} = 0 \tag{6b}$$

with

$$\psi_{;k} = \psi_{|k} - \Gamma_k\psi \tag{7a}$$

$$\bar{\psi}_{,k} = \bar{\psi}_{|k} + \bar{\psi}\Gamma_k \tag{7b}$$

and the four Γ_k matrices are defined by

$$\gamma_{i|k} - \{{}^{\,l}_{ik}\}\gamma_l - \Gamma_k\gamma_i + \gamma_i\Gamma_k = 0 \tag{8}$$

up to an imaginary multiple of the unit matrix. From this it follows that Γ_k can be chosen as

$$\Gamma_k = -\tfrac{1}{4}\gamma^j(\gamma_{j|k} - \gamma_l\{{}^{\,l}_{jk}\}) + \mathrm{i}eA_k \tag{9}$$

where A_k are the components of the electromagnetic potential. Noether's theorem applied to equation (4) gives the current density j^k and the matter tensor T_{ij}:

$$j^k = \mathrm{i}e\bar{\psi}\gamma^k\psi \tag{10}$$

$$T_{ij} = \tfrac{1}{4}[\bar{\psi}(\gamma_i\psi_{;j} + \gamma_j\psi_{;i}) - (\bar{\psi}_{;i}\gamma_j + \bar{\psi}_{;j}\gamma_i)\psi] \tag{11}$$

which have the usual conservation properties and

$$\mathrm{Tr}(T_i^j) = T_i^i = m\bar{\psi}\psi. \tag{12}$$

The generalized Dirac matrices γ^i are defined by

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij} \tag{13}$$

up to unitary transformations in Clifford space. We take:

$$\begin{aligned} \gamma_1 &= e^{\lambda/2}\tilde{\gamma}_1 & \gamma^1 &= e^{-\lambda/2}\tilde{\gamma}_1 \\ \gamma_2 &= r\tilde{\gamma}_2 & \gamma^2 &= r^{-1}\tilde{\gamma}_2 \\ \gamma_3 &= r\sin\theta\tilde{\gamma}_3 & \gamma^3 &= r^{-1}\sin^{-1}\theta\tilde{\gamma}_3 \\ \gamma_4 &= e^{\nu/2}\tilde{\gamma}_4 & \gamma^4 &= -e^{-\nu/2}\tilde{\gamma}_4 \end{aligned} \tag{14}$$

where the $\tilde{\gamma}_i$ are the usual Dirac matrices in Minkowski space. Thus the charge density is

$$j^4 = e\, e^{-\nu/2}\psi^\dagger\psi. \tag{15}$$

The Dirac equation (6a) now reads

$$\left[\tilde{\gamma}_1\, e^{-\lambda/2}\left(\frac{\partial}{\partial r} + r^{-1} + \frac{\nu'}{4}\right) + \frac{\tilde{\gamma}_2}{r}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right) + \frac{\tilde{\gamma}_3}{r\sin\theta}\frac{\partial}{\partial\phi} - e^{-\nu/2}\tilde{\gamma}_4\left(\frac{\partial}{\partial x^4} - ieA_4\right)\right]\psi - m\psi = 0 \tag{16}$$

from which we get the radial equations by substituting and separating variables using

$$\chi = e^{\nu/4} r(\sin\theta)^{1/2}\psi. \tag{17}$$

We remark that the operator K,

$$K = \tilde{\gamma}_1\tilde{\gamma}_4\tilde{\gamma}_2\frac{\partial}{i\,\partial\theta} + \tilde{\gamma}_1\tilde{\gamma}_4\tilde{\gamma}_3\frac{\partial}{i\sin\theta\,\partial\phi}, \tag{18}$$

is the generalized operator $K = \beta(\boldsymbol{\sigma}\cdot\boldsymbol{l} + 1)$ known from Dirac theory in Minkowski space with the usual integer eigenvalues ($K\chi = \kappa\chi$). Moreover, in a static gravitational field it is sufficient to consider only stationary solutions, so that the radial part of (16) can be regarded as a two-component spinor:

$$\chi = R(r)\Theta(\theta, \phi)\exp(-iEx^4); \qquad R(r) = \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} \tag{19}$$

which obeys the radial equations

$$\begin{aligned} \frac{dg(r)}{dr} &= -e^{\lambda/2}\frac{\kappa}{r}g(r) + e^{\lambda/2}[e^{-\nu/2}(E - V) + m]f(r) \\ \frac{df(r)}{dr} &= e^{\lambda/2}\frac{\kappa}{r}f(r) - e^{\lambda/2}[e^{-\nu/2}(E - V) - m]g(r) \end{aligned} \tag{20}$$

if we choose $\tilde{\gamma}_1$ and $\tilde{\gamma}_4$ to be

$$\tilde{\gamma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{\gamma}_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{21}$$

$V(r)$ is the Coulomb potential of the black hole and is given by

$$V(r) = eQ/r. \tag{22}$$

3. The Dirac (Klein–Gordon) equation in the field of an extended gravitating liquid drop

Since, classically, a periodic motion of particles in the field of a black hole is only possible outside the event horizon, one normally restricts the manifold M to the region outside the coordinate singularities of (2) or (3). Then one is able to interpret the theory within the frame of an asymptotic observer.

In order to understand better the results of the field equations in the restricted manifold, we now turn to the case without an event horizon which limits the solution of the Dirac (Klein–Gordon) equation and consider the Dirac (Klein–Gordon) equation in the field of an extended gravitating source (liquid drop) of radius r_0. For the metric we take (Adler *et al* 1965)

$$ds^2 = \begin{cases} \left(1-\frac{r^2}{\hat{R}^2}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) \\ \quad -\left[\frac{3}{2}\left(1-\frac{r_0^2}{\hat{R}^2}\right)^{1/2} - \frac{1}{2}\left(1-\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2 (dx^4)^2 \qquad r \leqslant r_0 & (23) \\ \left(1-\frac{2M_G}{r}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) \\ \quad -\left(1-\frac{2M_G}{r}\right)(dx^4)^2 \qquad r > r_0 & (24) \end{cases}$$

with $\hat{R}^2 = r_0^3/2M_G$. The radius r_0 of the drop should be greater than its Schwarzschild radius. Indeed, the model already breaks down for $r_0 = \frac{9}{8}r_S$, because of an infinite pressure in the matter tensor. In order to study the process $r_0 \to r_S$, we have analytically continued the metric to the case $r_S < r_0 \leqslant \frac{9}{8}r_S$

$$\begin{aligned} &e^{\nu} = a e^{br^2} && r \leqslant r_0 \\ &b = r_0^{-2}\left(\frac{r_0}{M_G}-2\right)^{-1} && a = \left(1-\frac{2M_G}{r_0}\right) e^{-br_0^2}. \end{aligned} \tag{25}$$

Note that for $r_0 \leqslant \frac{9}{8}r_S$, this metric does not account for a reasonable physical situation but should only serve as a definition of an energy eigenvalue of a field equation in this region†. By means of a Hamming modified predictor corrector method (HMPC) the radial Dirac equations (20) have been integrated numerically and the energy eigenvalues have been obtained by a generalization of a procedure due to Cohen (1960). With a given energy value E_0 we integrate (20) from both sides to a matching point r_m, where we match the f component to be continuous and find a corrected energy eigenvalue E_1 according to

$$E_1 = \frac{\langle\psi|\mathscr{H}|\psi\rangle}{\langle\psi|\psi\rangle} = E_0 + \frac{f(r_m)\Delta g(r_m)\, e^{\nu/2}(r_m)}{\langle\psi|\psi\rangle} = E_0 + \Delta E. \tag{26}$$

Here the Hamiltonian $\mathscr{H}$ stands for

$$\mathscr{H}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix} \equiv \frac{\kappa}{r} e^{\nu/2}\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} + m e^{\nu/2}\begin{pmatrix} g(r) \\ -f(r) \end{pmatrix} + e^{(\nu-\lambda)/2}\begin{pmatrix} -f'_{(r)} \\ g'_{(r)} \end{pmatrix}. \tag{27}$$

† Note, that the form of the energy spectrum obtained later might depend on the special analytic continuation. Physically meaningful forms of limiting procedures to the Schwarzschild geometry (e.g. in the context of gravitational collapse) will be investigated in a forthcoming article by the authors.

The successive iterations converge rapidly. The results are shown in figure 1 for the Dirac equation. For comparison we calculated the same quantities for the Klein–Gordon equation (Rafelski *et al* 1977). The results are nearly identical since we expect the spin–orbit coupling effects to be negligible. It should be noticed that the energy eigenvalues of all bound states (with finite numbers of nodes in their radial functions) tend to zero as $r_0 \to r_S$ and a quasi-continuum arises. We remark that the energy eigenvalues contain the average red-shift in the form of a factor $\langle\psi|e^{\nu/2}|\psi\rangle$ indicating the results obtained (Papapetrou 1956). In figure 1 we have defined continuum states as those which are asymptotically free, i.e. $|E| \geqslant mc^2$.

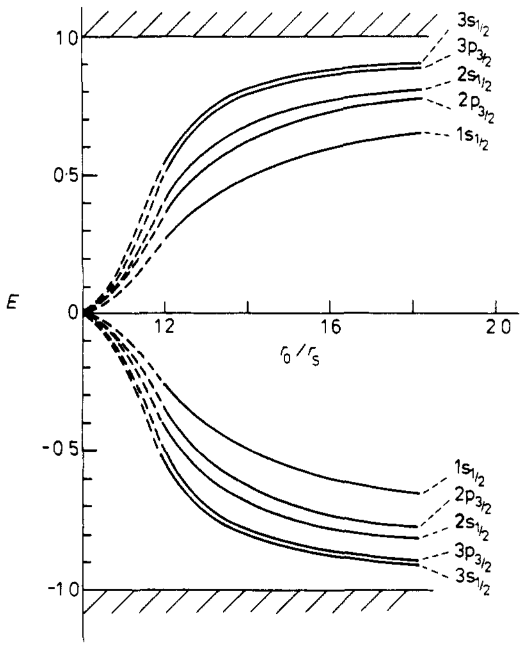

Figure 1. The energy eigenvalues of the Dirac equation in the field of an extended gravitating source. The energies are shown as a function of the source radius r_0 (in units of the Schwarzschild radius r_S).

4. The solution of the Dirac equation in a Schwarzschild (Reissner–Nordstrøm) geometry

In the case of the extended source (incompressible fluid model) we obtain the quasi-continuum as $r_0 \to r_S$ and vanishing energy eigenvalues of all bound states. So in the case of a coordinate singularity we expect to obtain a continuum with infinitely many nodes in the radial functions for energies between zero and mc^2. This is shown here for the Dirac equation. If the black hole is charged, we require that

$$Q_G^2 \leqslant M_G^2 \tag{28}$$

in order to have an event horizon. For a black hole of solar mass $M_\odot \approx 2 \times 10^{33}$ g, this means that $Q \approx 1{\cdot}7 \times 10^{20}$ C. The coordinate singularities lie at

$$r_\pm = M_G \pm (M_G^2 - Q_G^2)^{1/2}. \tag{29}$$

We then restrict our manifold by introducing the r^* coordinate (see equation (3)):

$$\mathrm{d}r^*/\mathrm{d}r = \Omega^{-2}. \tag{30}$$

The introduction of the r^* coordinate means that the outer coordinate singularity (event horizon) r_+ is projected to $-\infty$. The wavefunctions in the interior of a black hole, i.e. for $r \leq r_+$, are therefore not defined. By this procedure one avoids the introduction of boundary conditions for the wavefunctions at $r = r_+$.

The radial Dirac equations (20) in r^* coordinates are simply

$$\begin{aligned} \frac{\mathrm{d}g(r^*)}{\mathrm{d}r^*} &= -\frac{\kappa}{r}\Omega g(r^*) + [(E-V)+\Omega m]f(r^*) \\ \frac{\mathrm{d}f(r^*)}{\mathrm{d}r^*} &= \frac{\kappa}{r}\Omega f(r^*) - [(E-V)-\Omega m]g(r^*) \end{aligned} \tag{31}$$

which have as solution a plane wave for $r^* \to -\infty$, namely

$$\begin{aligned} g(r^*) &\simeq A_0 \sin[(E-\Lambda)r^* - \delta(E)] \\ f(r^*) &\simeq A_0 \cos[(E-\Lambda)r^* - \delta(E)]. \end{aligned} \tag{32}$$

Here $\Lambda \equiv eQ/r_+$ is the Coulomb potential of the black hole at its outer coordinate singularity r_+. Equation (31) has been solved numerically with the HMPC method. By means of (30) it has been integrated in r^* coordinates from large positive to negative values and we looked for resonances similar to those encountered in the case of the Klein–Gordon equation. For convenience the resonance parameter P_r which is the ratio of the first ($r^* \to +\infty$) amplitude to the resonance amplitude ($r^* \to -\infty$) of the large component in equation (31) has been introduced. The inverse of P_r is a measure for the probability that the resonance state decays towards the physical singularity, so resonances can be found by maximum values of P_r. P_r as a function of E can also be used to determine the width of a resonance. Typical wavefunctions of resonating and non-resonating type are depicted in figure 2. The results are in good agreement with those given in Deruelle and Ruffini (1974) and the energies of the resonances as well as the radial density distributions of the electron can be understood with the help of the effective potential. The effective potential was first derived from the classical Hamilton–Jacobi formalism by Christodoulou and Ruffini (1971). For spin-$\frac{1}{2}$ particles it must be re-derived by transforming the Dirac equation (31) into a second-order differential equation and applying the WKB approximation. The result is the effective potential, which for the Reissner–Nordstrøm field reads:

$$V_{\mathrm{eff}}^{\pm}(r) = \frac{eQ}{r} \pm \Omega\left(m^2 + \frac{\kappa^2}{r^2}\right)^{1/2}. \tag{33}$$

For the model of the extended gravitating source it is

$$V_{\mathrm{eff}}^{\pm}(r) = \pm \mathrm{e}^{\nu/2}\left(m^2 + \frac{\kappa^2}{r^2}\right)^{1/2}. \tag{34}$$

One finds that pair creation in the case of a charged black hole is due to the decay of the vacuum in analogy to the problems of strong fields in QED. This is demonstrated in figure 3. Even in the case of a neutral black hole the gap between the positive and negative energy continuum (particle and antiparticle states) is narrowed by the attractive gravitational interaction and vanishes for $r = r_S$. Due to the charge conjugation

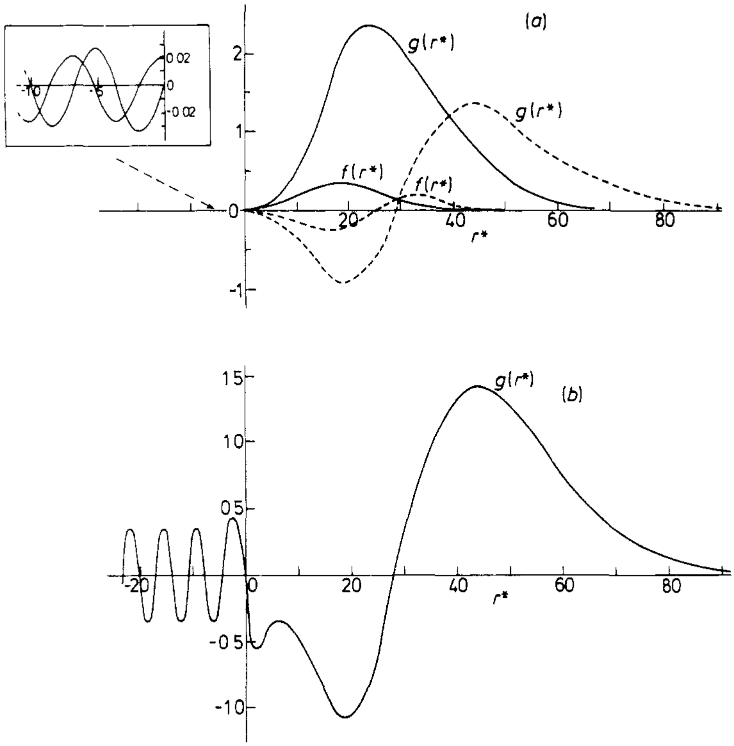

Figure 2. (*a*) The wavefunction for the first (full curves, $E = 0{\cdot}9761m$) and the second (broken curves, $E = 0{\cdot}9833m$) resonances of the Dirac equation in the Schwarzschild metric. Parameters: $M_G = 1$, $\kappa = 4$. (*b*) The large component $g(r^*)$ of a typical wavefunction off resonance. The extremum at $r^* \approx 6$ occurs at the innermost of the two classical turning points, i.e. $E = V_{\text{eff}}(r)$. At this point df/dr^* and dg/dr^* can vanish simultaneously to produce the extraordinary bump. At resonance wavefunctions the bound and the continuum parts join smoothly in phase.

invariance of the Schwarzschild field figure 3(*a*) is symmetric with respect to $E = 0$. This is caused by the tensorial character of the pure gravitational force which is quite similar to the effect of a scalar interaction.

With the introduction of a charged centre the symmetry between the electron and positron states is broken. Speaking in the language of hole theory, the occupied negative continuum is raised by a negatively charged centre and vice versa. If the Fermi energy is above $m_e c^2$ spontaneous electron emission (for negatively charged central objects), see figure 3(*b*), or spontaneous positron emission (for positively charged centres) takes place. This manifests the phase transition to a charged electron–positron vacuum. In other words pair creation occurs if $\Lambda = eQ/r_+ > m_e c^2$ which leads us to a limiting stable charge of a black hole, namely

$$Z_{\text{lim}} e^2/r_+ = m_e c^2. \tag{35}$$

In contrast to the same phenomenon in superheavy atoms, the right-hand side of (35) is only $m_e c^2$ and not $2m_e c^2$. Effectively only one particle must be created because the antiparticle is absorbed by the black hole. Since Z_{lim} is much smaller than the limit

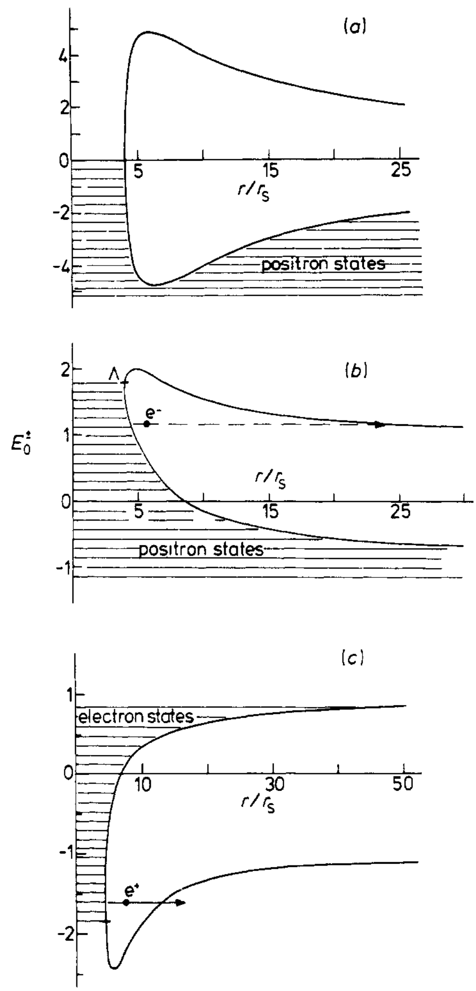

Figure 3. (*a*) The effective potential for the Schwarzschild field ($M_G = 2$, $\kappa = 50$). (*b*) The effective potential for the Reissner–Nordstrøm field ($Q_G = 1000$, $M_G = 2$, $\kappa = 4$). Pair creation occurs for $|\Lambda| \geq m_e c^2$. The centre is negatively charged and the decay of the vacuum happens via electron emission. (*c*) If the centre is positively charged ($Q_G = -1000$, $M_G = 2$, $\kappa = 10$) positrons are emitted for $|\Lambda| \geq m_e c^2$. The horizontal lines indicate the states of the upper continuum.

from equation (28), we can write, with $r_+ = 2GM/c^2$,

$$Z_{\text{lim}} = 2GMm_e/e^2. \tag{36}$$

For black holes of one solar mass this means $Z_{\text{lim}} \approx 10^{18}$ or a limiting charge of $C_{\text{lim}} \approx 0{\cdot}16$ C. This corresponds to a limiting charge-to-mass ratio per atom in a black hole of

$$\left(\frac{Ze}{M}\right) = \frac{2Gm_e}{e} = \frac{2Gm_e^2}{e^2}\left(\frac{e}{m_e}\right) = 0{\cdot}48 \times 10^{-42}\left(\frac{e}{m_e}\right) = 8{\cdot}4 \times 10^{-35}\ \text{C}\,\text{g}^{-1} \tag{37}$$

reflecting the double ratio of the gravitational ($\gamma = Gm_e^2/\hbar c$) to the electromagnetic ($\alpha = e^2/\hbar c$) coupling constant.

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