TABLE III. Values of $\bar{U}_2$ [Mev$\times(10^{-13}$ cm$)^3$] as a function of the parameters γ, λ for a Hulthén $n-p$ function, using a product wave function and δ-function potentials.

$\lambda \diagdown \gamma$	0.20	0.25	0.30	0.35
2.0	742	540	441	390
2.5	652	487	415	372
3.0	600	459	400	360
3.5	547	442	389	350

allowed with the use of δ-function potentials. In order to show this, a calculation has been carried through using a Hulthén form $e^{-\gamma r}(1-e^{-2\lambda\gamma r})/r$ for the deuteron core, for various values of γ and λ. The n-p potential was chosen to be of Yukawa form with the parameters previously given. The Λ-function $g(s)$ was obtained by direct integration of the Schrödinger equation; for this it was most convenient to consider the equation

$$u''+Be^{-x}(1-e^{-\lambda x})^2u/x^2=\eta^2u,$$

the eigenvalue $B(\eta,\lambda)$ being computed[23] as a function of η and λ. The value of the volume integral $\bar{U}_2(\gamma,\lambda)$ appropriate to $B_\Lambda=0.3$ Mev is readily obtained from this as a function of γ and λ. Table III gives values of $\bar{U}_2(\gamma,\lambda)$ in the relevant region and shows that $\bar{U}_2(\gamma,\lambda)$ decreases monotonically with increasing γ or λ. No local minimum appears (this can be shown analytically for the case $\lambda=0$); this is in contradiction with the result of the hypertriton calculation reported by Brown and Peshkin.[24]

In conclusion, it should be emphasized here that, although the use of δ-function potentials (without distortion of the nuclear core) allowed a convenient discussion of the qualitative features of the Λ-nucleon interaction from the binding energies of light hypernuclei,[6] the use of potentials with physically appropriate ranges is essential not only for the case of the hypertriton but generally for complex hypernuclei if quantitatively reliable estimates of the interaction strength are to be obtained.

[23] These numerical computations were performed at the Computing Center of Brookhaven National Laboratory, and we are pleased to thank Dr. M. Rose and Mr. P. Mumford for their assistance.

[24] L. M. Brown and M. Peshkin, Phys. Rev. **107**, 272 (1957). (*Note added in proof.*—Dr. Peshkin has informed us that he has recently carried out more accurate calculations with the product wave function, which agree with the above remarks in giving no local minimum for the case of δ-function potentials.)

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Past-Future Asymmetry of the Gravitational Field of a Point Particle

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The analytic extension of the Schwarzschild exterior solution is given in a closed form valid throughout empty space-time and possessing no irregularities except that at the origin. The gravitational field of a spherical point particle is then seen not to be invariant under time reversal for any admissible choice of time coordinate. The Schwarzschild surface $r=2m$ is not a singularity but acts as a perfect unidirectional membrane: causal influences can cross it but only in one direction. The apparent violation of the principle of sufficient reason seems similar to that which is associated with instabilities in other nonlinear phenomena.

I. INTRODUCTION

To define a gravitational universe we must give an analytic manifold and an analytic quadratic form $g_{\mu\nu}$ of correct signature.

For the manifold $\mathfrak{M}$ associated with a universe containing one point particle, one takes all of 4-space $\{x^\mu\}$ less the line $x^i=0$. (Greek indices$=0$, 1, 2, 3; Latin$=1$, 2, 3.) We might subject the gravitational field $g_{\mu\nu}(x)$ to the following requirements:

(a) The free space equation of Einstein:

$$R_{\mu\nu}(x)=0. \tag{1.1}$$

(b) Invariance under the one-parameter group

$$T_t:\begin{cases} x^0\rightarrow\bar{x}^0=x^0-t, \\ x^i\rightarrow\bar{x}^i=x^i. \end{cases} \tag{1.2}$$

(c) Invariance under the connected three-parameter group

$$R_r:\left\{\begin{matrix} x^0\rightarrow\bar{x}^0=x^0 \\ x^i\rightarrow\bar{x}^i=r_j{}^ix^j \end{matrix}\right\},\quad r^Tr=1,\quad \det r=1. \tag{1.3}$$

(d) Asymptotic to the Lorentz metric

$$g_{\mu\nu}(x)\rightarrow\eta_{\mu\nu},\quad x^ix^i\rightarrow\infty. \tag{1.4}$$

(e) Not extendable to the line $x^i=0$ (true singularity). This excludes the trivial case $g_{\mu\nu}(x)\equiv\eta_{\mu\nu}$.

(f) Invariant under the discrete group generated by

$$P:\begin{cases} x^0\rightarrow\bar{x}^0=x^0 \\ x^i\rightarrow\bar{x}^i=-x^i. \end{cases}$$

(g) Invariant under the discrete group generated by

$$T:\begin{cases} x^0 \to \bar{x}^0 = -x^0 \\ x^i \to \bar{x}^i = \quad x^i. \end{cases}$$

(h) x^0 timelike:

$$g_{00}(x) > 0 \quad \text{throughout } \mathfrak{M}. \tag{1.5}$$

If a solution to these conditions exists, it is given in part by Schwarzschild's line element,[1] but it is well known that this is analytic throughout $\mathfrak{M}$ only for a particle of negative mass. For positive mass the problem of analytic extension arises.

We shall see that for a particle of positive mass, the conditions (abcde) uniquely determine an analytic $g_{\mu\nu}(x)$. This $g_{\mu\nu}(x)$ then satisfies (f); but not (g) which is thus incompatible with (abcde). Likewise, (h) is not satisfied and is incompatible with (abcde).

Thus it seems that T invariance and general relativity are incompatible for a spherical point particle: although the requirements (abcde) do not distinguish between past and future, the only universe which obeys them does.

How is it possible that causes which are symmetric can have effects that are not? Such a violation of the principle of sufficient reason must be attributed to the nonlinear nature of gravitational theory. The time-symmetric universe can be extended from its asymptotic behavior to the gravitational radius and there it becomes singular. This wandering singularity is typical of nonlinear theories.[2] However, an analytic extension of the manifold through the gravitational radius is possible and is given in closed form below. The remarkable fact is that the extension is possible in exactly two distinct ways, neither of which is T invariant, but each of which is the other's image under T.

There must be theoretical processes in which this situation causes an instability of the type common in nonlinear phenomena. One imagines, for example, a slow process of accretion or compression that makes an initially weak gravitational source approach and finally pass through the critical situation in which the Schwarzschild radius of the source equals the extension of the source. At first the structure of the surrounding metric field is approximately time-symmetric. The present result suggests that near the critical situation the metric field will "buckle" into one of two quite distinct situations which are approximate T images of each other.

The Einstein-Rosen two-sheeted prolongation of the Schwarzschild exterior solution[3] now appears as a rather patchwork sort of manifold. Since the manifold $\mathfrak{M}$ with the metric (2.1) given below is analytic and complete and contains the manifold defined by the Schwarzschild exterior solution as a proper subset, it is the unique completion of the Schwarzschild exterior solution. One recalls of course that unlike $\mathfrak{M}$ the Einstein-Rosen manifold fails to satisfy the free-field equations at the "seam." Also the modified equations proposed by Einstein and Rosen[3] are satisfied by our solution as well as by the Einstein-Rosen solution and thus are deficient in an important uniqueness requirement.

II. LINE ELEMENT OF A POINT PARTICLE

On the manifold $\mathfrak{M}$, define the line-element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ by

$$ds^2 = (1-r^{-1})(dx^0)^2 + 2r^{-1}dx^0 dr - (1+r^{-1})dr^2 - (dx^i dx^i - dr^2). \tag{2.1}$$

Here the abbreviations $r = (x^i x^i)^{\frac{1}{2}}$, $dr = r^{-1}x^i dx^i$ were convenient. (It will be seen that the units used are such that the Schwarzschild gravitational radius and c are unity.) Note that this $g_{\mu\nu}$, analytic in $\mathfrak{M}$, has determinant

$$\det g \equiv 1 \quad \text{in } \mathfrak{M}. \tag{2.2}$$

Let us verify conditions (a)–(e) of Sec. I.

(a) For $r > 1$, the coordinate transformation

$$\bar{x}^0 = x^0 + \ln(r-1), \quad \bar{x}^i = x^i, \tag{2.3}$$

is well defined, and transforms the line element to

$$ds^2 = (1-\bar{r}^{-1})(d\bar{x}^0)^2 - (1-\bar{r}^{-1})^{-1}d\bar{r}^2 - (d\bar{x}^i d\bar{x}^i - d\bar{r}^2). \tag{2.4}$$

This is the Schwarzschild line element[1] and thus $\bar{R}_{\mu\nu} = 0$, $\bar{r} > 1$. Since $R_{\mu\nu}$ is a tensor it follows that $R_{\mu\nu} = 0$ for $r > 1$. Since $R_{\mu\nu}$ is analytic on $\mathfrak{M}$, it follows that $R_{\mu\nu} = 0$ everywhere on $\mathfrak{M}$.

(b), (c), (d) These are clear.

(e) This is shown by the singularity of the Petrov[4] scalars

$$\alpha_1 = \alpha_2 = 1/2r^3, \quad \alpha_3 = -1/r^3.$$

It is well known that the conditions (a)–(e) uniquely determine the solution except for scale. We now examine the remaining properties (f)–(h):

(f) This is clear.

(g) It is not sufficient to observe that "time"-reflection affects the sign of the term $\sim dx^0 dr$ to establish a past-future asymmetry. One would have to show that no coordinate transformation can eliminate this cross-term (globally). It suffices, however, to observe the following invariant distinction between the two halves of the light-cone at any point x^μ of $\mathfrak{M}$: One

[1] K. Schwarzschild, Sitzber. preuss. Akad. Wiss. Physik.-math. Kl. (1916), p. 189.

[2] See, for example, D. Finkelstein, Ph.D. thesis, Massachusetts Institute of Technology, 1953 (unpublished). See also N. V. Mitzkevich, Zhur. Eksptl. i Teort. Fiz. **29**, 354 (1955) [translation: Soviet Phys. JETP **2**, 197 (1956)] for similar results.

[3] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

[4] A. Z. Petrov, Sci. Mem. Kazan State Univ. **114**, 55 (1954); F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957). These scalars are eigenvalues of the complete curvature $R_{\kappa\lambda\mu\nu}$, regarded as defining a quadratic form on bivectors: $R(X,Y) = R_{\kappa\lambda\mu\nu}X^{\kappa\lambda}Y^{\mu\nu}$.

half-cone contains a null ray-direction which reaches the singular point $r=0$ when prolonged geodesically, the other half-cone does not. Indeed $n^\mu=-(r,x^1,x^2,x^3)$ is the only null ray-direction at (x^0,x^1,x^2,x^3) whose geodesic reaches $r=0$. The "time"-reflected direction Tn^μ (see Fig. 1) defines a geodesic which does not reach $r=0$ but instead becomes asymptotic to the surface $r=1$. In the manifold $\mathfrak{M}$ with the line-element (2.1) causal influences propagating into the "future" (increasing x^0) can cross the Schwarzschild surface only in an outward direction. This is seen immediately from an inspection of Fig. 1, which shows the light-cones at various points in the (x^0,x^1)-plane.

The surface $r=1$ is thus a true unidirectional membrane: causal influences can pass through it only in one sense. This again demonstrates the asymmetry between past and future.

Moreover, this demonstrates the existence of two distinct completions of the Schwarzschild exterior solution as asserted. One has the structure (2.1) and

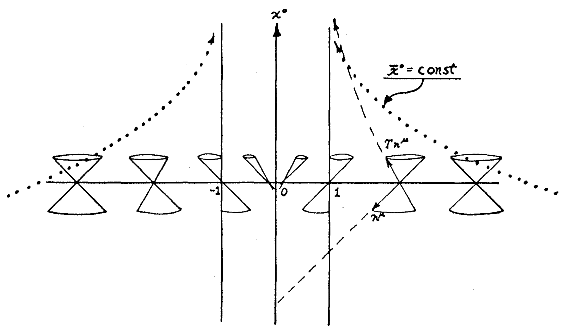

FIG. 1. The line-element in the (x^0,x^1)-plane. The light-cones in the (x^0,x^1,x^2)-space are indicated in perspective. The semipermeable membrane is $x^1=\pm 1$. A null geodesic (with tangent n^μ) is shown which reaches the center $x^1=0$. A null geodesic (with tangent Tn^μ) is shown which becomes asymptotic to the Schwarzschild surface. The lines of constant $\bar{x}^0$, defined in (2.3) and usually identified with astronomical time, are the dotted curves.

the other, obtained by time-reversal, has a negative coefficient of dx^0dr. The completions must be distinct even if the resulting manifolds are isomorphic under T, because a particular geodesic segment in one completion reaches the center and in the other completion does not.

(h) This is clear.

A transformation similar to (2.3) shows that the structure of the "core" $r<1$ of the manifold $\mathfrak{M}$ is that of the Schwarzschild interior solution. The only new information in this work, accordingly, is the connection of the interior and exterior solutions to form a single manifold.

So much for the one-body problem in general relativity. In conclusion, some speculations on the many-body problem: consider a universe populated with point singularities of positive mass. It is customary to assume that when they are far apart their short-range

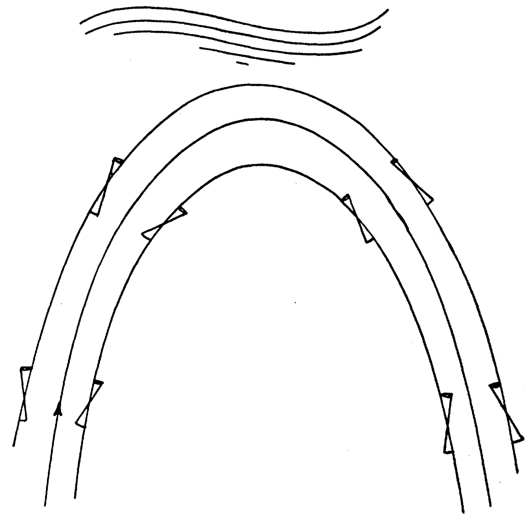

FIG. 2. Particle and antiparticle? If solutions to the free-space equations exist with a singular world-line like that shown schematically, then the two classes of particles are related as particle and antiparticle.

structures are little affected by each other. Then each must surround itself with a unidirectional surface, similar to the surface $r=1$ of (2.1). Accordingly they all fall into two classes according to whether their surfaces are permeable inwards or outwards, one class being the time-reversal of the other. It may be that according to the gravitational equations universes like Fig. 2 can exist. If so, the two classes of particles constitute a classical model of the relation between a particle and its antiparticle. On the other hand, in view of the delicate nature of the choice between the two classes, it is possible that the gravitational equations imply that all particles in one universe belong to the same class. If that is the case, the law of interaction between the particles is apparently asymmetric with respect to interchange of past and future, in spite of their origin in general relativity.†

III. ACKNOWLEDGMENTS

The result presented here is a by-product of a topological program in which I have received valuable aid from Charles W. Misner. The possibility of extending the Schwarzschild metric in a nonstatic manner was also studied by John J. Stachel, who discussed his work with me. Frequent discussions with James L. Anderson, Bruce Crabtree, and Melvin Hausner have provided important stimulus for this work.

† *Note added in proof.*—Schild points out that $\mathfrak{M}$ is still *incomplete*, it possesses a nonterminating geodesic of finite length in one direction. Kruskaal has sketched for me a manifold $\mathfrak{M}^*$ that is complete and contains $\mathfrak{M}$. $\mathfrak{M}^*$ is time-symmetric and violates one of the conditions on $\mathfrak{M}$: it does not have the topological structure of all of 4-space less a line. Kruskaal obtained $\mathfrak{M}^*$ some years ago (unpublished).